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THE PROPERTIES OF NONOSCILLATION AND FINITE VALENCE

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1. Introduction.

In this paper we shall consider the differential equation

$$y''(z) + p(z)y(z) = 0, \qquad |z| < 1, \tag{1.1}$$

where $p(z)$ is a regular function in the open unit circle E. The ratio $f(z) = u(z)/v(z)$ of any two independent solution $u(z)$ and $v(z)$ of (1.1) will be a function $f(z)$, meromorphic in E with only simple poles, and such that $f'(z) \neq 0$. The Schwarizian derivative of $f(z)$,

$$S_f(z) = \varphi'_f(z) - \frac{1}{2}\varphi_f^2(z), \ \varphi_f(z) = f''(z)/f'(z)$$

is connected with $p(z)$ by

$$S_f(z) = 2p(z). \tag{1.2}$$

If no solution of (1.1) (except the solution $y(z) = 0$) has more than one zero in E then $f(z)$ is univalent in E. Conversely, every univalent function $f(z)$ in E can be written as the ratio of two independent solutions of the equation (1.1). These connections were first stated by Z.Nehari ([1] Theorem I). In this paper we give that the connections of nonoscilation and finite valence. In Section 2, Theorem 2.1 may be state us a criteria of nonossilation. In Section 3, we obtain a simpler criteria for the finite valent of single valent meromorphic function.

2. A criteria of nonoscillation.

(1.1) is called nonoscillation in E if none of its solutions (except $y(z) = 0$) has infinite many zeros in E. Correspondingly we call a single valued meromorhic function finite valent in a domain D if for each a the equation $f(z) = a$ has only a finite number of solutions z in D.

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THEOREM 2.1. *Let $p(z)$ be regular in $|z| < 1$ and assume there exists x_0, $0 < x_0 < 1$, such that for all z with $x_0 < |z| < 1$*

$$(1 - |z|^2)^2 |p(z)| \le 1. \tag{2.1}$$

Then (1.1) is nonoscillation in $|z| < 1$.

Proof. Let $z_1, z_2 (z_1 \ne z_2)$ be any two points inside E. z_1, z_2 determine uniquely a circle C passing through them and orthogonal to $|z| = 1$. Let us call the part of C which lies between z_1 and z_2 and inside E, the orthogonal arc between z_1 and z_2, and let us denote it by $[z_1 z_2]$. Let x_0 be fixed and denote the ring $x_0 < |z| < 1$ by R.

Assume now that there exists a nontrivial solution $y(z)$ of (1.1) with infinitely many zeros in E. From this infinity of zeros we choose a sequence converging to a point α on $|z| = 1$. It follow that we can choose two zeros z_1 and z_2 of $y(z)$, belonging to this sequence, such that they, together with the orthogonal arc between them, lie in R.

There exists a linear transformation from $|z| < 1$ onto $|\zeta| < 1$ given by

$$z = e^{i\theta} \frac{\zeta - \alpha}{1 - \overline{\alpha}\zeta}, \qquad |\alpha| < 1, \tag{2.2}$$

which carries z_1 and z_2 into $\zeta = \rho$ and $\zeta = -\rho$ respectively $(0 < \rho < 1)$. (2.2) transforms $[z_1 z_2]$ into the segment $(-\rho, \rho)$. Define for $|\zeta| < 1$ by

$$g(\zeta) = f\left(e^{i\theta} \frac{\zeta - \alpha}{1 - \overline{\alpha}\zeta}\right). \tag{2.3}$$

The substitution (2.2) transforms (1.1) into

$$y_1''(\zeta) + p_1(\zeta) y_1(\zeta) = 0, \tag{2.4}$$

where

$$S_g(\zeta) = 2p_1(\zeta) \tag{2.5}$$

and

$$y\left(e^{i\theta} \frac{\zeta - \alpha}{1 - \overline{\alpha}\zeta}\right) = y_1(\zeta)\sigma(\zeta). \tag{2.6}$$

Here $\sigma(\zeta)$ is regular and nonzero in $|\zeta| < 1$. It follows that there exists a solution $y_1(\zeta) \neq 0$ of (2.4) such that $y_1(\rho) = y_1(-\rho) = 0$. Setting $\zeta = x + iy$, multiplying (2.4) on the segment $(-\rho, \rho)$ by $\overline{y}_1$ and integrating from $-\rho$ to ρ, we obtain

$$\int_{-\rho}^{\rho} |y_1'|^2 dx = \int_{-\rho}^{\rho} p_1 |y_1|^2 dx.$$

Writing $y_1 = u + iv$ we have

$$\int_{-\rho}^{\rho} (u_x^2 + v_x^2) dx = \int_{-\rho}^{\rho} p_1 (u^2 + v^2) dx. \tag{2.7}$$

It can be shown that (2.2) and (2.3) imply

$$|S_f(z)|(1 - |z|^2)^2 = |S_g(\zeta)|(1 - |\zeta|^2)^2.$$

It follows therefore by (1.2), (2.1) and (2.5) that

$$(1 - x^2)^2 |p_1(x)| \leq 1, \qquad -\rho \leq x \leq \rho.$$

Hence,

$$\left| \int_{-\rho}^{\rho} p_1 (u^2 + v^2) dx \right| \leq \int_{-\rho}^{\rho} \frac{u^2 + v^2}{(1 - x^2)^2} dx < \rho^2 \int_{-\rho}^{\rho} \frac{u^2 + v^2}{(\rho^2 - x^2)^2} dx.$$

Now the inequality

$$\rho^2 \int_{-\rho}^{\rho} \frac{u^2}{(\rho^2 - x^2)^2} dx < \int_{-\rho}^{\rho} u'^2 dx$$

holds for continuously differentiable real functions $u(x)$, $-\rho \leq x \leq \rho$, which have at $\pm\rho$ zeros of the first order [1]. Then we have

$$\left| \int_{-\rho}^{\rho} p_1 (u^2 + v^2) dx \right| < \int_{-\rho}^{\rho} (u_x^2 + v_x^2) dx,$$

which gives the desired contradiction to (2.7) and we have therefore proved Theorem 2.1. □

This nonoscillation Theorem may now be stated as a criteria of finite valent for meromorphic functions .

COROLLARY 2.1.1. *Let $f(z)$ be meromorphic in $|z| < 1$ and assume that*

$$(1-|z|^2)^2|S_f(z)| \leq 2 \text{ for } x_0 < |z| < 1,\ 0 < x_0 < 1$$

Then $f(z)$ is finite valent in $|z| < 1$.

Proof. Assume that there exists a complex number ω(which may be ∞) such that $f(z)-\omega = 0$ has an infinity of roots in E, Then there exist $z_1, z_2, (z_1 \neq z_2)$ such that $f(z_1) = f(z_2) = \omega$, and that z_1, z_2 and the orthogonal arc between them lie in R.

Consider now $f(z)$ and the corresponding (1.1) not in E, but only in any simply connected domain D containing the arc $[z_1 z_2]$ and contained in R. We obtain therefore a solution $y(z)$ of (1.1), analytic and therefore single valued in D, such that $y(z_1) = y(z_2) = 0$, while $p(z)$ satisfies (2.1) in D (and especially on $[z_1 z_2]$). But only this used in the proof of Theorem 2.1. □

NEHARI RESULT ([2]) : *For the unit circle he proved that if $p(z)$* is regular in $|z| < 1$ and if

$$\int_0^{2\pi} |p(e^{i\theta})| d\theta < \infty, \tag{2.8}$$

then (1.1) is nonoscillation.

The integral on the left hand side of (2.8) is defined as the limit, for $\rho \longrightarrow 1$, of the nondecreasing function

$$\int_0^{2\pi} |p(e^{i\theta})| d\theta$$

and (2.8) is therefore equivalent to

$$\int_0^{2\pi} |p(\rho e^{i\theta})| d\theta < c,\ c < \infty, 0 < \rho < 1. \tag{2.9}$$

Nehari Result may be deduced from Theorem 2.1. Indeed, setting

$$p(z) = \sum_{n=0}^{\infty} a_n z^n,$$

(2.9) implies

$$|a_n| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|p(\rho e^{i\theta})|}{\rho^{n+1}}\rho d\theta < \frac{c}{2\pi\rho^n},\ n = 0, 1, \ldots.$$

Letting $\rho \longrightarrow 1$, we obtain $|a_n| \leq c/2\pi$ and therefore

$$|p(z)| \leq \sum_{n=0}^{\infty} |a_n||z_n|^n \leq \frac{c}{2\pi(1-|z|)}.$$

This implies now the existence of at x_0, such (2.1) holds for $x_0 < |z| < 1$, i.e., the assumption of Theorem 2.1 is satisfied. So (1.1) is nonoscillation.

3. A finite valent theorem for a domain.

Let D be a simply connected domain in the z plane, having at least two boundary points and let $w = \psi(z)$ be a function mapping D onto $|w| < 1$. Let D' be any closed domain lying in the interior of D and denote by R' the domain $D - D'$. The map of R' under the transformation $w = \psi(z)$ covers a circular ring $R, x_0 < |w| < 1$, with $0 < x_0 < 1$ and x_0 near enough to 1. Let $f(z)$ be a meromorphic function in D and define $g(w)$ in $|w| < 1$ by

$$g(w) = f(\psi^{-1}(w)).$$

$f(z)$ is finite valent in D if and only if $g(w)$ is so in $|w| < 1$. The transformation formula $S_f(z)$ under the conformal mapping $w = \psi(z)$ is

$$[S_f(z) - S_\psi(z)] = S_g(w)\left(\frac{dw}{dz}\right)^2. \tag{3.1}$$

Applying now Corollary 2.1.1 to $g(w)$ it follows that $f(z)$ will be finite valent in D if the condition

$$|S_f(z) - S_\psi(z)| \leq \frac{2}{(1-|\psi(z)|^2)^2}\left|\frac{d\psi}{dz}\right|^2 \tag{3.2}$$

holds for all $z \in D - D'$. Similarly it follows that if $p(z)$ is regular in D and if

$$|p(z) - \frac{1}{2}S_\psi(z)| \leq \frac{1}{(1-|\psi(z)|^2)^2}\left|\frac{d\psi}{dz}\right|^2 \tag{3.3}$$

holds for all z in $D - D'$, then (1.1) is nonoscillation in D.

REMARK. (3.2) and (3.3) are independent of the normalization of the Riemann mapping function $w = \psi(z)$ mapping D onto $|w| < 1$. Let $w_1 = \psi_1(z)$ be another such function mapping D onto $|w_1| < 1$. The function $w_1(w) = \psi_1(\psi^{-1}(w))$ is a linear mapping of $|w| < 1$ onto $|w_1| < 1$ and it follows by the invariance of the Schwarzian derivative with respect to all linear transformation, that $S_{w_1}(z) = S_w(z)$, i.e,

$$S_{\psi_1}(z) = S_{\psi}(z). \tag{3.4}$$

Moreover, for a linear mapping of the unit circle onto itself, the relation

$$\frac{1 - |w_1(w)|^2}{1 - |w|^2} = \left|\frac{dw_1}{dw}\right|$$

holds, which implies

$$\frac{1}{(1 - |\psi_1(z)|^2)^2}\left|\frac{d\psi_1}{dz}\right|^2 = \frac{1}{(1 - |\psi(z)|^2)^2}\left|\frac{d\psi}{dz}\right|^2. \tag{3.5}$$

(3.4) and (3.5) show clearly that condition (3.2) and (3.3) are independent of the normalization of the mapping $\psi(z)$.

Restricting ourselves to domains bounded by a finite number of Jordan curves, we have the following property:

Let D be a multiply connected domain in the z plane, bounded by a finite number of Jordan curves. Let S be its universal covering surface. Let $w = \psi(z)$ map S onto $|w| < 1$ and let D' be any closed domain in D. A function $f(z)$, meromorphic and single valued in D, will be finitely valent there if condition (3.3) holds for all z in $D - D'$.

This property enable us now to obtain a simpler criterion for the finite valence of single valued meromorphic functions in the case in which the n-boundaries of the domain are analytic Jordan curves.

THEOREM 3.1. *Let D be a domain in the z-plane such that its boundary B consists of a finite number of analytic Jordan curves. Let S be its universal covering surface. Let $z_0 \in D$ and denote by B_ε the level curve $g(z, z_0, D) = \varepsilon$, $\varepsilon > 0$, of the harmonic Green's function $g(z, z_0, D)$ with pole at z_0. Let $w = \psi(z)$ map S onto $|w| < 1$ and $f(z)$ be meromorphic and single valued in D and set*

$$M(\varepsilon) = \max_{z \in B_\varepsilon} |S_f(z)|.$$

If

$$\lim_{\varepsilon\to 0}\varepsilon^2 M(\varepsilon)=0, \tag{3.6}$$

then $f(z)$ is finite valent in D.

Proof. Suppose D is not simply connected. Choose $\psi(z)$ on S so that $\psi(z_0)=0$. Let z be the coordinate in D and not on S, so that $\psi(z)$ is a many valued function. By the connecting the n - boundary curves $B_1,\ldots B_n$ of D by $n-1$ cuts $\nu_1,\ldots,\nu_{n-1}$, we obtain a simply connected domain D^*. D^* allows us to fix uniquely a branch $\psi_\iota(z)$ of $\psi(z)$. Assume that none of the cuts $\nu_1,\ldots,\nu_{n-1}$ go through z_0. Let the branch $\psi_\iota(z)$ be defined $\psi_\iota(z_0)=0$, and consider the behavior of this branch in D and on B. From the analyticity of the bounary curves it follows that $\psi_\iota(z)$ and its derivatives are piecewise analytic on B. Moreover, $\frac{d\psi_\iota(z)}{dz}\neq 0$ in $\overline{D}=D\cup B$ and it follows that, for all z in D,

$$|S_{\psi_\iota}(z)|=|S_\psi(z)|\leq M,\ 0<M<\infty, \tag{3.7}$$

$$\left|\frac{d\psi_\iota(z)}{dz}\right|\geq m,\ 0<m<\infty. \tag{3.8}$$

For every $\varepsilon>0$ let us now consider the following two closed region in D :

$$D_1(\varepsilon)=\{z: g(z,z_0,D)\geq\varepsilon\}$$

and

$$D_2(\varepsilon)=\{z|z=\psi^{-1}(w),\ |w|\leq e^{-\varepsilon}\}.$$

Then we have [5, pp. 50-51]

$$D_2(\varepsilon)\subset D_1(\varepsilon). \tag{3.9}$$

(3.9) implies now that $|\psi(z)|\geq e^{-\varepsilon}$ for the level curve $B_\varepsilon(g(z,z_0,D)=\varepsilon)$ and in particular

$$|\psi_\iota(z)|\geq e^{-\varepsilon},\ z\in B_\varepsilon,\varepsilon>0.$$

We have therefore for each $z \in B_\varepsilon$

$$1 - |\psi_i(z)|^2 = (1 + |\psi_i(z)|)(1 - |\psi_i(z)|) \leq 2(1 - e^{-\varepsilon}) < 2\varepsilon$$

which implies

$$M(\varepsilon)(1 - |\psi_i(z)|^2)^2 < 4M(\varepsilon)\varepsilon^2. \tag{3.10}$$

Using now our assumption (3.6), it follows from (3.1), (3.5), (3.7) (3.8) and (3.10) that there exists $\varepsilon_0 > 0$ such that

$$|S_f(z) - S_{\psi_i}(z)| \leq \frac{2}{(1 - |\psi_i(z)|^2)^2} \left| \frac{d\psi_i}{dz} \right|^2 .$$

for all z with $0 < g(z, z_0, D) < \varepsilon_0$, i.e., for all $z \in D - D_1(\varepsilon_0)$. So we have proved Theorem 3.1 for a multiply connected domain.

If D is simply connected domain we use condition (3.2). Relation (3.7) and (3.8) hold now for the single valued function $\psi(z)$ and in this case, clearly, $D_1(\varepsilon) = D_2(\varepsilon)$. Therefore Theorem 3.1 is established. □

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