Pusan Kyŏngnam Math. J. 11(1995), No 2, pp. 249–258

JACOBI OPERATORS IN 2–STEP NILPOTENT LIE GROUP WITH 1–DIMENSIONAL CENTER

KEUN PARK

1. Introduction

Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $\langle , \rangle$, and N its unique simply connected Lie group with the left invariant metric determined by the inner product $\langle , \rangle$ on $\mathcal{N}$. The meaning of $\mathcal{N}$ being 2-step nilpotent is $[\mathcal{N}, [\mathcal{N}, \mathcal{N}]] = 0$. The center of $\mathcal{N}$ is denoted by $\mathcal{Z}$. Then, $\mathcal{N}$ can be expressed as the direct sum of the subspaces $\mathcal{Z}$ and its orthogonal complement $\mathcal{Z}^{\perp}$.

For Z in $\mathcal{Z}$, a skew-symmetric linear transformation $j(Z) : \mathcal{Z}^{\perp} \to \mathcal{Z}^{\perp}$ is defined by $j(Z)X = (adX)^* Z$ for $X \in \mathcal{Z}^{\perp}$, or equivalently

$$\langle j(Z)X, Y \rangle = \langle [X, Y], Z \rangle \qquad \text{for} \quad X, Y \in \mathcal{Z}^{\perp}.$$

This transformation was defined by A. Kaplan[4,5] to study the geometry of groups of Heisenberg type, those groups for which $j(Z)^2 = -|Z|^2 id$ for each $Z \in \mathcal{Z}$.

The Jacobi operator plays a fundamental role in Riemannian geometry. In [1], it was showed that the Jacobi operator along each geodesic of groups of Heisenberg type has constant eigenvalues. The main purpose of this paper is to show that the Jacobi operator along each geodesic of a 2-step nilpotent Lie group with 1-dimensional center has constant eigenvalues.

Received October 17, 1995.

Supported in part by the Basic Sciences Research Institute of the University of Ulsan .

2. Preliminaries

In this section, we will give some known results about 2-step nilpotent Lie groups with a left invariant metric. Throughout this section, we denote $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $\langle , \rangle$, and N its unique simply connected Lie group with the left invariant metric induced by the inner product $\langle , \rangle$ on $\mathcal{N}$.

Recall that for $Z_0 \in \mathcal{Z}$, a skew-symmetric linear transformation $j(Z_0) : \mathcal{Z}^\perp \to \mathcal{Z}^\perp$ is defined by $\langle j(Z_0)X, Y\rangle = \langle [X,Y], Z_0\rangle$ for $X, Y \in \mathcal{Z}^\perp$. Let $\{\pm\theta_1 i, \pm\theta_2 i, \cdots, \pm\theta_n i\}$ be the distinct eigenvalues of $j(Z_0)$ with each $\theta_k > 0$, and let $\{W_1, W_2, \cdots, W_n\}$ be the invariant subspaces of $j(Z_0)$ such that $j(Z_0)^2 = -\theta_k^2 id$ on W_k for each $k = 1, 2, \cdots, n$. Then, $\mathcal{Z}^\perp$ can be expressed as a direct sum of W_k's, that is $\mathcal{Z}^\perp = Ker j(Z_0) \oplus \oplus_{k=1}^n W_k$ and $j(Z_0)^2 = -\theta_k^2 id$ on each W_k leads

$$e^{tj(Z_0)} = \cos(t\theta_k)id + \frac{\sin(t\theta_k)}{\theta_k} j(Z_0) \tag{2.1}$$

on W_k for each k.

Let $\gamma(t)$ be a curve in N such that $\gamma(0) = e$(identity element of N) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^\perp$ and $Z_0 \in \mathcal{Z}$. Since $\exp : \mathcal{N} \to N$ is a diffeomorphism, the curve $\gamma(t)$ can be expressed uniquely by $\gamma(t) = \exp(X(t) + Z(t))$ with

$$\begin{aligned} X(t) \in \mathcal{Z}^\perp, \quad X'(0) = X_0, \quad X(0) = 0 \\ Z(t) \in \mathcal{N}, \quad Z'(0) = Z_0, \quad Z(0) = 0. \end{aligned}$$

A.Kaplan[4,5] showed that the curve $\gamma(t)$ is a geodesic in N if and only if

$$\begin{aligned} X''(t) &= j(Z_0)X'(t), \\ Z'(t) + \frac{1}{2}[X'(t), X(t)] &\equiv Z_0. \end{aligned} \tag{2.2}$$

The solution to this equation was obtained by P. Eberlein(See [2]), and he obtained the following(See Propositin 3.2 [2]).

$$\gamma'(t) = dl_{\gamma(t)}(X'(t) + Z_0) \tag{2.3}$$

where $l_{\gamma(t)}$ is the left translation by $\gamma(t)$, and it is trivial that

$$X'(t) = e^{tj(Z_0)}X_0 \tag{2.4}$$

from Kaplan's equations (2.2).

Regarding elemets of $\mathcal{N}$ as left invariant vector fields on N, it is routine to show that

(2.5)

$$\nabla_X Y = \frac{1}{2}[X,Y] \qquad \text{for} \quad X, Y \in \mathcal{Z}^{\perp},$$
$$\nabla_X Z = \nabla_Z X = -\frac{1}{2}j(Z)X \qquad \text{for} \quad X \in \mathcal{Z}^{\perp}, Z \in \mathcal{Z},$$
$$\nabla_Z Z^* = 0 \qquad \text{for} \quad Z, Z^* \in \mathcal{Z}.$$

And also, from (2.5), the formulas for the curvature tensor given by

$$R(\xi_1,\xi_2)\xi_3 = -\nabla_{[\xi_1,\xi_2]}\xi_3 + \nabla_{\xi_1}(\nabla_{\xi_2}\xi_3) - \nabla_{\xi_2}(\nabla_{\xi_1}\xi_3)$$

are obtained as follows(See [2]).

(2.6)

$$R(X,Y)X^* = \frac{1}{2}j([X,Y])X^* - \frac{1}{4}j([Y,X^*])X + \frac{1}{4}j([X,X^*])Y$$
$$\text{for} \quad X, Y, X^* \in \mathcal{Z}^{\perp},$$
$$R(X,Y)Z = -\frac{1}{4}[X,j(Z)Y] + \frac{1}{4}[Y,j(Z)X]$$
$$R(X,Z)Y = -\frac{1}{4}[X,j(Z)Y] \quad \text{for} \quad X, Y \in \mathcal{Z}^{\perp} \text{and} \qquad Z \in \mathcal{Z},$$
$$R(Z,Z^*)X = -\frac{1}{4}j(Z^*)j(Z)X + \frac{1}{4}j(Z)j(Z^*)X$$
$$R(X,Z)Z^* = -\frac{1}{4}j(Z)j(Z^*)X \quad \text{for} \quad X \in \mathcal{Z}^{\perp} \quad \text{and} \quad Z, Z^* \in \mathcal{Z}$$
$$R(Z_1,Z_2)Z_3 = 0 \quad \text{for} \quad Z_1, Z_2, Z_3 \in \mathcal{Z}.$$

3. Main Results

Throughout this section, all matrices concerned are over the real numbers field R. For a square matrix A, we denote $|A|$ the determinant of A, $A(\alpha\underline{i})(A(\alpha\overline{i}))$ the matrix obtained by multiplying α to i-th row(column) in A, $A(\underline{i}+\alpha\underline{j})(A(\overline{i}+\alpha\overline{j}))$ the matrix obtained by adding the α times of j-th row (column) to the i-th row(column) in A and $A(i,j), A(i)$ in brief for $i=j$, the submatrix of A obtained by deleting the i-th row and j-th column in A. And also, we denote I the identity matrix.

NOTATIONS 3.1. *Let*
$\mathbf{x}=(p_1,-q_1,p_2,-q_2,\cdots,p_n,-q_n)\in R^{2n}$,
$\mathbf{y}=(\theta_1 q_1,\theta_1 p_1,\theta_2 q_2,\theta_2 p_2,\cdots,\theta_n q_n,\theta_n p_n)\in R^{2n}$ *and*
$A_{2n}=-3\,{}^T\mathbf{x}\,\mathbf{x}+diag(\theta_1^2,\theta_1^2,\theta_2^2,\theta_2^2,\cdots,\theta_n^2,\theta_n^2)$ *be* $2n\times 2n$ *matrix where* ^{T} *is the transpose and* $diag(\lambda_1,\lambda_2,\cdots,\lambda_m)$ *is a diagonal matrix of order* m *with the diagonals* $\lambda_1,\lambda_2,\cdots,\lambda_m$. *And let*

$$B_{2n+1}=\left(\begin{array}{c|c}\alpha & -\mathbf{y}\\ \hline -{}^T\mathbf{y} & A_{2n}\end{array}\right),\quad \text{where } \alpha=\sum_{k=1}^{n}(p_k^2+q_k^2)\,.$$

LEMMA 3.2. *Let each* θ_k *be a fixed real number in Notation 3.1. Then,*

(1) *all coefficients of polynomials* $|A_{2n}-xI|$ *and* $|A_{2n}(2n-1)-xI|+|A_{2n}(2n)-xI|$ *depend only on* $\{p_1^2+q_1^2,\cdots,p_n^2+q_n^2\}$, *and*

(2) *all coefficients of the polynomial* $|B_{2n+1}-xI|$ *depend only on* $\{p_1^2+q_1^2,\cdots,p_n^2+q_n^2\}$.

We will give the proof of Lemma 3.2 after Theorem 3.3.

THEOREM 3.3. *Let* N *be a simply connected 2-step nilpotent Lie group with a left-invariant metric. If* N *has 1-dimensional center, then the eigenvalues of Jacobi operator along each geodesic in* N *are constant.*

proof. Since N has a left invariant metric, it is sufficient to show the statement about geodesics $\gamma(t)$ with $\gamma(0)=e$ where e is the identity of N. Let $\gamma(t)$ be a geodesic in N with $\gamma(0) = e$ and $\gamma'(t) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^\perp$ and $Z_0 \in \mathcal{Z}$. Then, from (2.3), we have

$$(3.1) \qquad \begin{aligned} \gamma'(t) &= dl_{\gamma(t)}(X'(t) + Z_0) \\ &= X'(t) + Z_0 \end{aligned}$$

where the last terms are regarded as left invariant vector fields along $\gamma'(t)$.

Recall that the Jacobi operator along $\gamma(t)$ is

$$R_{\gamma'(t)}(\cdot) := R(\cdot, \gamma'(t))\gamma'(t).$$

From (2.6), we have the formula of Jacobi operator of 2-step nilpotent Lie group as follows.

$$(3.2) \qquad \begin{aligned} &R_{\gamma'(t)}(X + Z) \\ =&R_{X'(t)+Z_0}(X + Z) \\ =&\frac{3}{4}j([X, X'(t)])X'(t) + \frac{1}{2}j(Z)j(Z_0)X'(t) - \frac{1}{4}j(Z_0)j(Z)X'(t) \\ &- \frac{1}{4}j(Z_0)^2 X - \frac{1}{2}[X, j(Z_0)X'(t)] + \frac{1}{4}[X'(t), j(Z_0)X] \\ &+ \frac{1}{4}[X'(t), j(Z)X'(t)] \end{aligned}$$

for any $X \in \mathcal{Z}^\perp$ and $Z \in \mathcal{Z}$. Since N has 1-dimensional center, (3.2) can be reduced to

$$(3.3) \qquad \begin{aligned} &R_{X'(t)+Z_0}(X + Z) \\ =&\frac{3}{4}j([X, X'(t)])X'(t) + \frac{1}{4}j(Z)j(Z_0)X'(t) - \frac{1}{4}j(Z_0)^2 X \\ &+ \frac{1}{4}([X'(t), j(Z)X'(t)] - [X, j(Z_0)X'(t)]) \end{aligned}$$

for any $X \in \mathcal{Z}^\perp$ and $Z \in \mathcal{Z}$. If $Z_0 = 0$, then we see $X'(t) = X_0$ from (2.4), and so $R_{\gamma'(t)}(\cdot) = R_{X_0}(\cdot)$ has constant eigenvalues.

Suppose that $Z_0 \neq 0$. We may assume that $|Z_0| = 1$ by rescaling $|X_0|$ if necessary. From (3.3), it is easy to see that $R_{X'(t)+Z_0}(\cdot) = 0$ on

$Kerj(Z_0)$ and $R_{X'(t)+Z_0}(\cdot)$ is $Kerj(Z_0)^{\perp}$-invariant. So, we may also assume that $j(Z_0)$ is nonsingular. Let $\{\pm\theta_1 i, \pm\theta_2 i, \cdots, \pm\theta_n i\}$ be the distinct eigenvalues of $j(Z_0)$ with each $\theta_k > 0$, and let $\{W_1, W_2, \cdots, W_n\}$ be the invariant subspaces of $j(Z_0)$ such that $j(Z_0)^2 = -\theta_k^2 id$ on W_k for each $k = 1, 2, \cdots, n$. Since $\mathcal{Z}^{\perp} = \oplus_{k=1}^{n} W_k$, we can let $X_0 = \sum_{k=1}^{n} x_k E_k$ for some real numbers x_k's and $E_k \in W_k$ with $|E_k| = 1$. Hence, we get, from (2.1) and (2.4),

$$\begin{aligned}(3.4)\qquad X'(t) &= e^{tj(Z_0)}X_0 \\ &= \sum_{k=1}^{n} x_k e^{tj(Z_0)} E_k \\ &= \sum_{k=1}^{n} x_k\{\cos(t\theta_k)E_k + \frac{\sin(t\theta_k)}{\theta_k} j(Z_0)E_k\} \\ &= \sum_{k=1}^{n} (\xi_k E_k + \eta_k \bar{E}_k)\end{aligned}$$

where $\xi_k = x_k \cos(t\theta_k)$, $\eta_k = x_k \sin(t\theta_k)$ and $\bar{E}_k = \frac{1}{\theta_k} j(Z_0)E_k$ for $k = 1, 2, \cdots, n$. Note that $j(Z_0)$ is skew-symmetric. For each k, since $\{E_k, \bar{E}_k\}$ is orthonormal, it can be extended to an orthonormal basis β_k for W_k. Let $\beta = \{Z_0\} \cup \cup_{k=1}^{n} \beta_k$ be an orthonormal basis for $\mathcal{N}$. Then, direct calculations of $R_{X'(t)+Z_0}(\cdot)$ in (3.3) with (3.4) lead the following:

$$4R_{X'(t)+Z_0}(Z_0) = -\sum_{k=1}^{n} \theta_k^2(\xi_k E_k + \eta_k \bar{E}_k) + \sum_{k=1}^{n} \theta_k^2(\xi_k^2 + \eta_k^2) Z_0$$

$$4R_{X'(t)+Z_0}(E_k) = 3\theta_k \eta_k \sum_{i=1}^{n} \theta_i(-\eta_i E_i + \xi_i \bar{E}_i) + \theta_k^2 E_k - \theta_k^2 \xi_k Z_0$$

$$4R_{X'(t)+Z_0}(\bar{E}_k) = -3\theta_k \xi_k \sum_{i=1}^{n} \theta_i(-\eta_i E_i + \xi_i \bar{E}_i) + \theta_k^2 \bar{E}_k - \theta_k^2 \eta_k Z_0$$

$$4R_{X'(t)+Z_0}(E) = \theta_k^2 E \quad \text{for each} \quad k \quad \text{and} \quad E \in \beta_k - \{E_k, \bar{E}_k\}.$$

Hence, letting $p_k = \theta_k \eta_k$ and $q_k = \theta_k \xi_k$ for each k and reordering β by

$$\beta = \{Z_0, E_1, \bar{E}_1, \cdots, E_n, \bar{E}_n\} \cup \cup_{k=1}^{n} (\beta_k - \{E_k, \bar{E}_k\}),$$

we obtain the representation matrix G of $4R_{X'(t)+Z_0}(\cdot)$ with respect to the ordered basis β as follows:

$$\mathbf{G} = \left(\begin{array}{c|c} B_{2n+1} & 0 \\ \hline 0 & \Gamma \end{array}\right)$$

where B_{2n+1} is in Notation 3.1 and $\Gamma = diag(\theta_1^2, \cdots, \theta_1^2, \theta_2^2, \cdots, \theta_2^2, \cdots, \theta_n^2, \cdots, \theta_n^2)$. Therefore, by Lemma 3.2(2), we see that each coefficients of the characteristic polynomial of G depend only on $\{p_1^2 + q_1^2 = \theta_1^2 x_1^2, \cdots, p_n^2 + q_n^2 = \theta_n^2 x_n^2, \theta_1^2, \cdots, \theta_n^2\}$. This means that the eigenvalues of the Jacobi operator along $\gamma(t)$ are constant. In fact, the eigenvalues depend only on $\{\theta_1^2, \cdots, \theta_n^2\}$ and $\{x_1^2, \cdots, x_n^2\}$. This completes the proof.

Proof of Lemma 3.2. Without loss of generality, we may assume that each $p_k^2 + q_k^2 \neq 0$. First, we will show (1) by induction for n. For $n = 1$, (1) is obvious. Suppose that (1) holds for $n-1$. For each $k = 1, 2, \cdots, n$, let

$$\begin{aligned} C &= (c_{ij}) \\ &= (A_{2n} - xI)(p_k\underline{2k})(p_k\overline{2k})(\underline{2k} + q_k\underline{2k-1})(\overline{2k} + q_k\overline{2k-1}). \end{aligned}$$

Then, using expansions of $|C|$ with respect to $2k$-th row and $|C(2k, 2k-1)|$ with respect to $(2k-1)$-th column, we have

(3.5)

$$\begin{aligned} & p_k^2 |A_{2n} - xI| \\ &= |C| \\ &= -c_{(2k)(2k-1)} |C(2k, 2k-1)| + c_{(2k)(2k)} |C(2k)| \\ &= (p_k^2 + q_k^2)(\theta_k^2 - x)|A_{2n}(2k) - xI| - q_k^2(\theta_k^2 - x)^2 |A_{2n}(2k)(2k-1) - xI|. \end{aligned}$$

By applying the similar argument to

$$(A_{2n} - xI)(q_k\underline{2k-1})(q_k\overline{2k-1})(\underline{2k-1} + p_k\underline{2k})(\overline{2k-1} + p_k\overline{2k})$$

we obtain

(3.6)

$$q_k^2|A_{2n}-xI| =(p_k^2+q_k^2)(\theta_k^2-x)|A_{2n}(2k-1)-xI| \\ - p_k^2(\theta_k^2-x)^2|A_{2n}(2k)(2k-1)-xI|.$$

Adding (3.6) to (3.5) implies

(3.7)

$$|A_{2n}-xI| = (\theta_k^2-x)\{(|A_{2n}(2k)-xI|+|A_{2n}(2k-1)-xI|) \\ -(\theta_k^2-x)|A_{2n}(2k)(2k-1)-xI|\}$$

for $k=1,2,\cdots,n$.

And also, for each $k=1,2,\cdots,n-1$, by applying the similar arguments to

$$(A_{2n}(2n)-xI)(\alpha\underline{2n-1})(\alpha\overline{2n-1})(\underline{2n-1}+p_n\underline{2k-1})(\overline{2n-1}+p_n\overline{2k-1})$$

and

$$(A_{2n}(2n-1)-xI)(\alpha\underline{2n-1})(\alpha\overline{2n-1})(\underline{2n-1}+q_n\underline{2k})(\overline{2n-1}+q_n\overline{2k})$$

with $\alpha\in\{p_k,q_k\}$, we obtain

(3.8)

$$p_k^2|A_{2n}(2n)-xI| =\{p_k^2(\theta_n^2-x)+p_n^2(\theta_k^2-x)\}|A_{2n-2}-xI| \\ - p_n^2(\theta_k^2-x)^2|A_{2n-2}(2k-1)-xI|,$$

(3.9)

$$p_k^2|A_{2n}(2n-1)-xI| =\{p_k^2(\theta_n^2-x)+q_n^2(\theta_k^2-x)\}|A_{2n-2}-xI| \\ - q_n^2(\theta_k^2-x)^2|A_{2n-2}(2k-1)-xI|,$$

(3.10)

$$q_k^2|A_{2n}(2n)-xI| =\{q_k^2(\theta_n^2-x)+p_n^2(\theta_k^2-x)\}|A_{2n-2}-xI| \\ - p_n^2(\theta_k^2-x)^2|A_{2n-2}(2k)-xI|,$$

(3.11)
$$q_k^2|A_{2n}(2n-1)-xI| = \{q_k^2(\theta_n^2-x)+q_n^2(\theta_k^2-x)\}|A_{2n-2}-xI| - q_n^2(\theta_k^2-x)^2|A_{2n-2}(2k)-xI|.$$

Adding (3.9),(3.10) and (3.11) to (3.8) implies

(3.12)
$$\begin{aligned}&(p_k^2+q_k^2)(|A_{2n}(2n)-xI|+|A_{2n}(2n-1)-xI|)\\ =&2\{(p_k^2+q_k^2)(\theta_n^2-x)+(p_n^2+q_n^2)(\theta_k^2-x)\}|A_{2n-2}-xI|\\ &-(p_n^2+q_n^2)(\theta_k^2-x)^2(|A_{2n-2}(2k)-xI|+|A_{2n-2}(2k-1)-xI|).\end{aligned}$$

From (3.7) for $k=n$ and (3.12) for $k=n-1$, we get that (1) holds for n. This completes the proof of (1).

Next, we will prove (2). Letting $D=(d_{ij})=B_{2n+1}-xI$, we have that

(3.13)
$$\begin{aligned}&|B_{2n-1}-xI|\\ =&\sum_{j=1}^{2n+1}(-1)^{1+j}d_{1j}|D(1,j)|\\ =&\{\sum_{j=1}^{n}(p_j^2+q_j^2)-x\}|A_{2n}-xI|-\sum_{k=1}^{n}\theta_k(p_k|D(1,2k+1)|-q_k|D(1,2k)|).\end{aligned}$$

Expanding $|D(1,2k+1)(p_k\underline{2k})(\underline{2k}+q_k\underline{2k-1})|$ with respect to $2k$-th row, we obtain that

(3.14)
$$\begin{aligned}&p_k|D(1,2k+1)|\\ =&|D(1,2k+1)(p_k\underline{2k})(\underline{2k}+q_k\underline{2k-1})|\\ =&\theta_k(p_k^2+q_k^2)|A_{2n}(2k)-xI|+q_k(\theta_k^2-x)|D(1,2k+1)(2k,2k)|.\end{aligned}$$

Similarly, expanding $|D(1,2k)(q_k\underline{2k-1})(\underline{2k-1}+p_k\underline{2k})|$ with respect to $(2k-1)$-th row, we obtain that

(3.15)
$$\begin{aligned}&q_k|D(1,2k)|\\ =&|D(1,2k)(q_k\underline{2k-1})(\underline{2k-1}+p_k\underline{2k})|\\ =&-\theta_k(p_k^2+q_k^2)|A_{2n}(2k-1)-xI|-p_k(\theta_k^2-x)|D(1,2k)(2k-1,2k)|.\end{aligned}$$

And also, expanding

$|D(1,2k+1)(2k,2k)(q_k \underline{2k-1})|$ and $|D(1,2k)(2k-1,2k)(p_k \underline{2k-1})|$

with respect to $(2k-1)$-th row, we have that

(3.16)

$$
\begin{aligned}
&q_k|D(1,2k+1)(2k,2k)| + p_k|D(1,2k)(2k-1,2k)| \\
=&|D(1,2k+1)(2k,2k)(q_k \underline{2k-1})| + |D(1,2k)(2k-1,2k)(p_k \underline{2k-1})| \\
=&-\theta_k(p_k^2+q_k^2)|A_{2n}(2k)(2k-1)-xI|.
\end{aligned}
$$

Hence, from (3.14), (3.15) and (3.16), we see that

(3.17)

$$
\begin{aligned}
&p_k|D(1,2k+1)| - q_k|D(1,2k)| \\
=&\theta_k(p_k^2+q_k^2)(|A_{2n}(2k)-xI| + |A_{2n}(2k-1)-xI|) \\
&-(\theta_k^2-x)|A_{2n}(2k)(2k-1)-xI|.
\end{aligned}
$$

Substituding (3.17) to (3.13) and using (3.7), we have that

$$
|B_{2n+1}-xI| = -x\{1+\sum_{k=1}^{n}(p_k^2+q_k^2)(\theta_k^2-x)^{-1}\}|A_{2n}-xI|,
$$

which implies that $|B_{2n+1}-xI|$ depends only on $\{p_1^2+q_1^2,\cdots,p_n^2+q_n^2\}$ by (1).

References

1. J. Berndt, F. Tricerri and L. Vanhecke, *Geometry of generalized Heisenberg groups and their Damek-Ricci harmonic extensions*, C. R. Acad. Sci. Paris **318** (1994), 471–476.
2. P. Eberlein, *The Geometry of 2-step Nilpotent Lie groups with a left invariant metric*, Ann. Scient. Ecole Normale Sup. **27** (1994), 611–660.
3. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York, 1978.
4. A. Kaplan, *Riemannian nilmanifolds attached to Clifford modules*, Geom. Dedicata **11** (1981), 127–136.
5. ———, *On the geometry of the groups of Heisenberg type*, Bull. London Math. Soc. **15** (1983), 35–42.

Department of Mathematics
University of Ulsan
Ulsan 680-749, Korea