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k^*–PARANOMAL–OPERATORS

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1. Introduction

In [1], Masathosi Fujii and Yasuhiko Nakatsu introduced the concept of k-hyponomal operators. An operator T is said to be k-hypornomal if $(TT^*)^k \leq (T^*T)^k$, k being a positive integer.

The class of k-quasi-hyponormal operators and k-paranomal operators was introduced and studied by Rushpa R.Suri and N.Singh[2]. An operator T defined on a Hilbert space $\mathcal{H}$ is k-quasi-hyponomal if $T^{*k}T^k - (T^*T)^k \geq 0$, k being a positive integer, or equivalently $\|(T^*T)^{\frac{k}{2}}x\| \geq \|T^kx\|$, if k is even, and $\|T^kx\|$, if k is odd, for all x in $\mathcal{H}$. An operator T is said to be k-paranomal if $\|Tx\|^k \leq \|T^kx\|$ for every unit vector x in $\mathcal{H}$, k being a positive integer. In [3], an operator T is said to be $*$-paranomal if $\|T^*x\|^2 \leq \|T^2x\|$ for every unit vector x in $\mathcal{H}$.

We shall define a new class of operators. An operator T is said to be k^*-paranomal if $\|T^*x\|^k \leq \|T^kx\|$ for every unit vector x in $\mathcal{H}$, k being a positive integer. It is the aim of this note to introduce a new class of operator which generalizes the class of $*$-paranomal operators, and to give some properties. Our new class of k^*-paranomal operators occcupies the following place;

$$\{\text{Nomal}\} \subset \{k-\text{hypernomal}\} \subset \{k^*-\text{paranomal}\} \subset \{\text{Nomaloid}\}.$$

Various examples have been constructed to show the proper inclusion relation among the classes.

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2. k^*-Paranomal Opertors

THEOREM 1. *Let T be a weighted shift operator with weights $\{\alpha_n\}$, then T is k^*-paranomal if and only if $|\alpha_{n-1}|^k \leq |\alpha_n||\alpha_{n+1}|\cdots|\alpha_{n+k-1}|$ for all integer n.*

Proof. This follows immediately from the definition of k^*-paranomal.

LEMMA 2 [1]. *For each k, if T is k-hyponomal, then T^k is hyponomal operator.*

THEOREM 3. *If T is k-hyponomal, then T is k-quasi-hyponomal.*

Proof. Note that T is 1-hyponomal if and only if T is hyponomal. We shall prove inductively that

$$(T^*T)^k \leq T^{*k}T^k.$$

Suppose that they are true for $k = n-1$ and T is n-hyponomal. Then we have, by Lemma 2

$$(T^*T)^n = T^*(TT^*)^{n-1}T \leq T^*(T^*T)^{n-1}T \leq T^*(T^{*n-1}T^{n-1})T = T^{*n}T^n.$$

Therefore T is n-quasi-hyponomal, which complete the proof.

THEOREM 4. *For $2 \leq k$, if T is k-hypornomal, then T is k^*-paranomal.*

Proof. Ths is clear by Theorem 2.4[2],

$$\|Tx\|^k \leq \|(T^*T)^{\frac{k}{2}}x\| \leq \|T^kx\|,$$

if k is even, and

$$\|Tx\|^k \leq \|T(T^*T)^{\frac{k-1}{2}}x\| \leq \|T^kx\|,$$

if k is odd, for each unit vector x in $\mathcal{H}$. Hence the inequality $\|T^*x\|^k \leq \|T^kx\|$ holds for each unit vector x in $\mathcal{H}$, and so T is k^*-paranomal.

COROLLARY 5. *If T is k-hyponormal, then T is k-paranomal.*

REMARK 6. *A $*$-paranomal need not be a k-hyponormal operator.*

EXAMPLE 1. *Let $\mathcal{K}$ be the direct sum of denumerable copies of $\mathcal{H}$. Let A and B be any positive operators on $\mathcal{H}$. Let n be any fixed positive integer, define an operator $T = T_{A.B.n}$ on $\mathcal{K}$ as $T_{A.B.n}(x_1.x_2, \cdots) = (0, Ax_1, \cdots, Ax_n, Bx_{n+1}, Bx_{n+2}, \cdots)$. A simple computation shows that the operator $T = T_{A\ B.n}$ is k-hyponormal if and only if $0 \leq B^{2k} - A^{2k}$. Let $T = T_{A\ B,n}$, where A and B are positive operators on $\mathcal{H}$ satisfying this time $A^2 = C$ and $B^4 = D$, where C and D are positive operators on $\mathcal{H}$ defined as*

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}.$$

Then T is $*$-paranomal operator if and only if $B^4 - 2\lambda A^2 + \lambda^2 \geq 0$ for each $\lambda > 0$. And

$$B^4 - 2\lambda A^2 + \lambda^2 = D - 2\lambda C + \lambda^2 = \begin{pmatrix} (1-\lambda)^2 & 2(1-\lambda) \\ 2(1-\lambda) & (2-\lambda)^2 + 4 \end{pmatrix},$$

which is a positive operator for each $\lambda > 0$. Hence T is $*$-paranormal operator. *However,*

$$B^8 - A^8 = D^2 - C^4 = \begin{pmatrix} -25 & -51 \\ -25 & -50 \end{pmatrix}$$

is not positive. Hence T is not a k-hypornomal operator.

THEOREM 7. *For $k \geq 3$, there exists a k^*-paranomal operator which is not $*$-paranomal operator.*

EXAMPLE 2. *For $k \geq 4$, let $\{e_n\}_{n=-\infty}^{n=\infty}$ be an orthonomal basis of the Hilbert space $\mathcal{H}$.*

Define a bilateral weighted shift T on $\mathcal{H}$ with $\{\alpha_n\}$ given by

$$\alpha_n = \begin{cases} \frac{1}{2} & \text{if } n \leq -1, \\ \sqrt{\frac{2}{5}} & \text{if } n = 0, \\ \frac{n}{n+1} & \text{if } n \geq 1. \end{cases}$$

Clearly $|\alpha_{n-1}|^k \leq |\alpha_n| \cdots |\alpha_{n+k-1}|$ for $n \neq 1$. For $n = 1$, $|\alpha_0|^k \leq |\alpha_1||\alpha_2| \cdots |\alpha_k|$. Therefore, it follows from Theorem 1 that T is k^*-paranomal. But T is not a $*$-paranormal operator as $\alpha_0 > \alpha_1$.

THEOREM 8. *For $k \geq 2$, there exists a k^*-paranomal operator which is not k-hyponomal operator.*

EXAMPLE 3. *For $k \geq 3$, let $\{e_n\}_{n=-\infty}^{n=\infty}$ be an orthonomal basis of the Hilbert space $\mathcal{H}$.*

Let T be a weight shift with weights α_n,

$$\alpha_n \begin{cases} \frac{1}{2} & \text{if } n \leq -1, \\ 2 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ 2 & \text{if } n = 2, 3, 4, \cdots, \end{cases}$$

Then T is 5^*-paranomal, but T is not 5-hyponomal.

THEOREM 9. *If T is a k^*-paranomal operator,then T is nomaloid.*

Proof. By Definition of $\|T\|$, i.e. $\|T\| = sup\{\|Tx\|; \|x\| = 1\}$, there exits a sequence $\{x_n\}$ of unit vectors such that $\|T^*x_n\| \to \|T^*\| = \|T\|$. We may suppose, without loss of generality that $\|T\| = 1$. Since for any unit vector x $\|T^*x\|^k \leq \|T^kx\|$, we have $\lim \|T^kx_n\| = 1$. This leads to $\lim \|T^jx_n\| = 1$, $1 \leq j \leq k$.

Since

$$\begin{aligned} \|T^{k+1}\| &= \|T^k \frac{Tx}{\|Tx\|}\| \|Tx\| \\ &\geq \|T^*Tx\|^k \frac{\|Tx\|}{\|Tx\|}, \end{aligned}$$

if we put in this inequality, $x = x_n$ we obtain

$$\lim \|T^{k+1}x_n\| = 1.$$

Also, if $\lim \|T^px_n\| = 1, p \leq l$, then $\lim \|T^{p+1}x_n\| = 1$.

This is the consequence of the inequality as follows :

$$\begin{aligned} \|T^{l+1}x\| &= \|T^kT^{l+1-k}x\| \\ &= \|T^k \frac{T^{l+1-k}x}{\|T^{l+1-k}x\|}\| \|T^{l+1-k}x\| \\ &\geq \|T^*T^{l+1-k}x\|^k \frac{\|T^{l+1-k}x\|}{\|T^{l+1-k}x\|^k} \\ &= \|T^*T^{l+1-k}x\|^k \|T^{l+1-k}x\|^{1-k} \end{aligned}$$

and

$$\begin{aligned}
\|T^*T^{l+1-k}x\| &= \|T^*TT^{l-k}x\| \\
&= \|T^*T\frac{T^{l-k}x}{\|T^{l-k}x\|}\|\|T^{l-k}x\| \\
&\geq \|T\frac{T^{l-k}x}{\|T^{l-k}x\|}\|^2\|T^{l-k}x\| \\
&= \frac{\|T^{l+1-k}x\|^2\|T^{l-k}x\|}{\|T^{l-k}x\|^2}.
\end{aligned}$$

Hence

$$\|T^{l+1}x\| \geq \|T^{l+1-k}x\|^{2k}\|T^{l+1-k}x\|^{1-k}\|T^{l-k}x\|^{-k}.$$

Therefore

$$\lim \|T^{l+1}x_n\| = 1.$$

By an introduction argument, we obtain that for every j

$$\lim \|T^j x_n\| = 1.$$

This proves the theorem.

From the above Examples and Theorems, and in [2], the class of k^*-paranomal operators finds the following place in the hierarchy of classes of operators as follows:

nomaloid

$\nearrow$ $\nwarrow$

k-paranomal k^*-paranomal

$\uparrow$

k-quasi-hyponomal

$\nwarrow$ $\uparrow$

k-hyponomal

where the symble $\rightarrow$ indicates the inclusion relation.

COROLLARY 10. *If T is a k^*-paranomal operator, then the spectral radius of T; $R_{sp}(T)$ is equal to $\|T\|$.*

References

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