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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS II

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1. Introduction

Let H be a real Hilbert space. We consider the initial value problem

$$\text{(E)}\qquad \begin{aligned} &\frac{du(t)}{dt} + Au(t) + G(u)(t) \ni f(t), \quad 0 < t < \infty, \\ &u(0) = x, \end{aligned}$$

where A is a maximal monotone (possibly multivalued) operator defined on a subset $D(A)$ contained in H, $x \in \overline{D(A)}$, $f \in L^1_{loc}([0,\infty) : H)$ and G is a given mapping

$$\text{(1.1)}\qquad G : C([0,T] : \overline{D(A)}) \longrightarrow L^1([0,T] : H), \text{ for all } T > 0.$$

Problems of the type (E) have been considered by many authors (see [7, 13, 14, 15]). Crandall and Nohel [7] obtained the existence result of solutions of (E) in connection with the study of a related nonlinear Volterra equation. Recently, Aizicovici [1] investigated nice asymptotic results of solutions of (E), which are the natural analogs of the evolution case (i.e., $G \equiv 0$). Mitidieri [14] studied the strong convergence of solutions of (E). In particular, by using the properties of almost nonexpansive curve which was introduced by Djafari Rouhani [8], Jung, Park and Kang [12] established the weak convergences of solutions of (E) and the Cesàro mean of solutions of (E).

The purpose of this paper is to study the strong convergence of the Cesàro mean of solutions of (E). Our study can be viewed as a

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continuation of [12]. In Section 2, we describe the notations and contain some definitions and known results. Section 3 contains the asymptotic properties of almost odd nonexpansive curve. Finally, in Section 4, we give a main asymptotic result of the Cesàro mean of solutions of (E).

2. Preliminaries

Let H be a real Hilbert space with inner product $(\ ,\ )$ and norm $\|\ \|$. Let A be a maximal monotone (possibly multivalued) operator defined on subset $D(A) \subset H$. As usual, we will put $[x, y] \in A \Leftrightarrow y \in Ax$. For background material concerning maximal monotone operators, see [3, 4].

We will use "$w - \lim$" or "$\rightharpoonup$" to indicate weak convergence in H. The symbol $\overline{D}$ denotes the closure of the set D. For a function $u : [0, \infty) \to H$, we denote by $\omega_w(u(t))$ the weak ω-limit set of u, i.e.,

$$\omega_w(u(t)) = \{y \in H : y = w - \lim u(t_n), \text{ for some sequence } t_n \to \infty\}$$

and by $\overline{co}\omega_w(u(t))$ the closed convex hull of $\omega_w(u(t))$, respectively. Let $u : [0, \infty) \to H$ be a bounded function. With the function $u(t)$, we associate the functional

$$\phi(y) = \limsup_{t \to \infty} \|u(t) - y\|^2.$$

Then ϕ is a continuous, strictly convex function on H, satisfying $\phi(y) \to \infty$ as $\|y\| \to \infty$, and therefore ϕ has a unique minimum in H. The unique point $c \in H$ satisfying

$$\phi(c) = \min_{y \in H} \phi(y)$$

is called the asymptotic center of $u(t)$ and it is denoted by $c = AC(u(t))$. For these facts, see [10].

Consider now the initial value problem (E), where G satisfies (1.1), $x \in \overline{D(A)}$ and $f \in L^1_{loc}([0, \infty) : H)$. We recall the following definitions [1, 7].

DEFINITION 2.1. A strong solution of (E) on $[0,\infty)$ is a function $u \in W^{1,1}_{loc}([0,\infty) : H) \cap C([0,\infty) : \overline{D(A)})$, satisfying $u(0) = x$ and $du(t)/dt + Au(t) + G(u)(t) \ni f(t)$, a.e. on $(0,\infty)$.

DEFINITION 2.2. A function $u \in C([0,\infty) : \overline{D(A)})$ is said to be a generalized solution to equation (E) if there are sequences $x_n \in \overline{D(A)}$, $f_n \in L^1_{loc}([0,\infty) : H)$ and $u_n \in C([0,\infty) : H)$ such that u_n is a strong solution of

$$\begin{aligned} &\frac{du_n}{dt} + Au_n + G(u_n) \ni f_n, \\ &u_n(0) = x_n, \end{aligned}$$

$x_n \to x$, $f_n \to f$ in $L^1([0,T] : H)$ and $u_n \to u$ in $C([0,T] : H)$, for each $0 < T < \infty$.

The following existence result is well-known [1, 7].

PROPOSITION 2.3. *Let G satisfy* (1.1) *and assume that:*

(i) *There exists $\gamma \in L^1_{loc}([0,\infty) : \mathbb{R})$ such that for every $u, v \in C([0,\infty) : \overline{D(A)})$,*

$$\|G(u) - G(v)\|_{L^1([0,t]:H)} \le \int_0^t \gamma(s)\|u - v\|_{L^\infty([0,s]:H)}ds, \quad 0 \le s \le t < \infty. \tag{2.1}$$

(ii) *For each $T \in (0,\infty)$, there is $\alpha_T : [0,\infty) \to [0,\infty)$ such that if $u \in C([0,T] : \overline{D(A)})$ is of bounded variation and $\|u\|_{L^\infty([0,T]:H)} \le R$, then*

$$var(G(u) : [0,t]) \le \alpha_T(R)(1 + var(u : [0,t])), \quad 0 \le t \le T \tag{2.2}$$

and

$$\|G(u)(0^+)\| \le \alpha_T(R).$$

Then,

(a) *For each $x \in D(A)$ and $f \in BV_{loc}([0,\infty) : H)$, problem* (E) *has a unique strong solution defined on $[0,\infty)$.*

(b) *For each $x \in \overline{D(A)}$ and $f \in L^1_{loc}([0,\infty) : H)$, problem* (E) *has a unique generalized solution defined on* $[0,\infty)$.

3. Asymptotic behavior of curves in H

In this section, we study asymptotic behavior of almost odd nonexpansive curve.

Let $u \in C([0,\infty) : H)$; in the sequel we refer to such u as a curve in H. Let $\sigma(t) = (1/t)\int_0^t u(\tau)d\tau$. We begin with the following:

DEFINITION 3.1 [8]. The curve $u(t)$ is almost nonexpansive (abbreviated ANEC) if for any $r, s, h \geq 0$,

$$\|u(r+h) - u(s+h)\|^2 \leq \|u(r) - u(s)\|^2 + \epsilon(r,s),$$

where

$$\lim_{r,s\to\infty} \epsilon(r,s) = 0.$$

DEFINITION 3.2. The curve $u(t)$ is almost odd nonexpansive (abbreviated AONEC), if curve $u(t)$ is an ANEC and for any $r, s, h \geq 0$,

$$\|u(r+h) + u(s+h)\|^2 \leq \|u(r) + u(s)\|^2 + \epsilon(r,s),$$

where

$$\lim_{r,s\to\infty} \epsilon(r,s) = 0.$$

REMARK 3.3. (a) A nonexpansive curve $\{u(t)\}$ that satisfies

$$\|u(r+h) - u(s+h)\| \leq \|u(r) - u(s)\|$$

for any $r, s, h \geq 0$, is an ANEC.

(b) A bounded curve $\{u(t)\}$ that satisfies

$$\|u(r+h) - u(s+h)\| \leq \|u(r) - u(s)\| + \epsilon_1(r,s)$$

for any $r, s, h \geq 0$, where $\lim_{r,s\to\infty} \epsilon_1(r,s) = 0$, is an ANEC.

In our next results, we will use the following notation:

$$E(u(t)) = \{q \in H : \lim_{t\to\infty} \|u(t) - q\| \text{ exists}\}.$$

Note that if $E(u(t)) \neq \emptyset$, then curve $\{u(t)\}$ is bounded.

We need the following lemmas, which are given in [8, 12].

LEMMA 3.4 [8]. *Let $\{u(t)\}$ be an ANEC. Then*

$$\omega_w(\sigma(t)) \subset E(u(t)).$$

LEMMA 3.5 [12]. *Let $\{u(t)\}$ be an ANEC in H. Then the following are equivalent:*

(i) $w-\lim_{t\to\infty}\sigma(t)$ *exists.*

(ii) $E(u(t)) \neq \emptyset$. *Moreover, if $w-\lim_{t\to\infty}\sigma(t)$ exists, then it is the asymptotic center of $\{u(t)\}$.*

THEOREM 3.6. *Let $\{u(t)\}$ be any curve in H. Then the following are equivalent:*

(i) $(u(r), u(r+h)) \to \alpha(h)$ *as $r \to \infty$ uniformly in $h \geq 0$ (we call this the property $(*)$).*

(ii) $\{u(t)\}$ *is an AONEC.*

Proof. (i) $\Rightarrow$ (ii): We have $\|u(t)\|^2 \to \alpha(0)$ as $t \to \infty$ and hence $\{u(t)\}$ is a bounded curve in H and for all $\epsilon > 0$ there exists $l > 0$ such that

$$|\ (u(r+h), u(s+h)) - (u(r), u(s))\ | < \epsilon$$

for all $r,\ s \geq l,\ h \geq 0$. Therefore by taking

$$\begin{aligned}\epsilon(r,s) &= |\ \|u(r+h) - u(s+h)\|^2 - \|u(r) - u(s)\|^2\ | \\ &\leq |\ \|u(r+h)\|^2 + \|u(s+h)\|^2 - \|u(r)\|^2 - \|u(s)\|^2\ | \\ &\quad + 2|\ (u(r+h), u(s+h)) - (u(r), u(s))\ |,\end{aligned}$$

we get

$$\limsup_{r,s\to\infty} \epsilon(r,s) \leq 2\epsilon, \text{ uniformly in } h \geq 0.$$

This implies that $\{u(t)\}$ is an ANEC in H, since $\epsilon > 0$ was arbitrary. Similarly, from the equality

$$\begin{aligned}&\|u(r+h) + u(s+h)\|^2 - \|u(r) + u(s)\|^2 \\ &= \|u(r+h)\|^2 + \|u(s+h)\|^2 - \|u(r)\|^2 - \|u(s)\|^2 \\ &\quad + 2((u(r+h), u(s+h)) - (u(r), u(s))),\end{aligned}$$

it follows that $\|u(r+h)+u(s+h)\|^2 - \|u(r)+u(s)\|^2 \to 0$ as $r,\ s \to \infty$ (uniformly in $h \geq 0$). Therefore $\{u(t)\}$ is an AONEC in H.

(ii) $\Rightarrow$ (i): First we prove that it follows from (ii) that $\|u(t)\|$ converges. Taking $r = s$ and using the oddness, we get

$$4\|u(r+h)\|^2 \leq 4\|u(r)\|^2 + \epsilon_1(r).$$

Hence for all $r \geq 0$,

$$\limsup_{s\to\infty} \|u(s)\|^2 \leq \|u(r)\|^2 + \frac{\epsilon_1(r)}{4}.$$

Thus

$$\limsup_{s\to\infty} \|u(s)\|^2 \leq \liminf_{r\to\infty} \|u(r)\|^2,$$

which implies that $\|u(t)\|$ converges. Now let us prove that $(u(s), u(s+h))$ is Cauchy, uniformly in $h \geq 0$. In fact, we have for $l \leq k$,

$$\|u(h+k) - u(k)\|^2 \leq \|u(h+l) - u(l)\|^2 + \epsilon(h+l, l).$$

From this it follows that

$$\begin{aligned} &2[(u(h+l), u(l)) - (u(h+k), u(k))] \\ &\leq \|u(h+l)\|^2 + \|u(l)\|^2 - \|u(h+k)\|^2 - \|u(k)\|^2 + \epsilon(h+l, l). \end{aligned}$$

Similarly, from the inequality

$$\|u(h+k) + u(k)\|^2 \leq \|u(h+l) + u(l)\|^2 + \epsilon_1(h+l, l),$$

it follows that

$$\begin{aligned} &2[(u(h+k), u(k)) - (u(h+l), u(l))] \\ &\leq \|u(h+l)\|^2 + \|u(l)\|^2 - \|u(h+k)\|^2 - \|u(k)\|^2 + \epsilon_1(h+l, l). \end{aligned}$$

Therefore we have

$$\begin{aligned} &2|\ (u(h+l), u(l)) - (u(h+k), u(k))\ | \\ &\leq \|u(h+l)\|^2 + \|u(l)\|^2 - \|u(h+k)\|^2 - \|u(k)\|^2 \\ &\quad + \max(\epsilon(h+l, l), \epsilon_1(h+l, l)) \xrightarrow[l \leq\ k]{l,\ k\to\infty} 0 \end{aligned}$$

uniformly in $h \geq 0$ since $\|u(t)\|$ converges, and this complete the proof.

As a immediate result, we have the following:

COROLLARY 3.7. *Let $\{u(t)\}$ be any curve in H satisfying the property $(*)$. Then $\sigma(t)$ converges weakly to some $p \in H$, where p is the asymptotic center of the curve $\{u(t)\}$. Moreover we have $(u(t), p) \to \|p\|^2$ as $t \to \infty$.*

Proof. By Theorem 3.6, $\{u(t)\}$ is an AONEC, and hence it is an ANEC. And since $\|u(t)\|$ is convergent, it is bounded. Then, since Hilbert space is reflexive, $\{\sigma(t)\}$ has a subnet $\{\sigma(t_n)\}$ which converges weakly to some $p \in H$. By Lemma 3.4, $\omega_w(\sigma(t)) \subset E(u(t))$, and so $p \in E(u(t))$. If there exists another subnet $\{\sigma(t_l)\}$ which converges weakly to some $q \in H$, then we also have $q \in E(u(t))$. Hence the net

$$2(u(t), q-p) + \|p\|^2 - \|q\|^2 = \|u(t)-p\|^2 - \|u(t)-q\|^2$$

has a limit as $t \to \infty$, i.e., $\lim_{t\to\infty}(u(t), q-p)$ exists. Therefore $(p, q-p) = (q, q-p)$, which implies $\|p-q\|^2 = 0$, and hence $p = q$. Hence every weakly convergent subnet of $\sigma(t)$ converges weakly to p, and hence $w - \lim_{t\to\infty}\sigma(t) = p$, which is the asymptotic center of $\{u(t)\}$ by Lemma 3.5. Then, by Lemma 3.4, we also have $p \in E(u(t))$, and hence $\lim_{t\to\infty}\|u(t)-p\|^2 = \delta^2$ exists. But we have

$$\begin{aligned}(u(t), p) &= \frac{1}{2}(\|u(t)\|^2 + \|p\|^2 - \|u(t)-p\|^2)\\ &\xrightarrow[t\to\infty]{} \frac{1}{2}(\alpha(0) + \|p\|^2 - \delta^2).\end{aligned}$$

Thus $\lim_{t\to\infty}(u(t), p)$ exists, and since $(\sigma(t), p) \to \|p\|^2$ as $t \to \infty$, it follows that $\lim_{t\to\infty}(u(t), p) = \|p\|^2$. This completes the proof.

Now we denote the several notations;

$$d = \lim_{t\to\infty}\|u(t)\|,$$

$$\rho(r) = \lim_{s\to\infty}\|u(r) - u(s)\|, \quad \rho^*(r) = \lim_{s\to\infty}\|u(r) + u(s)\|,$$

$$a(t) = \frac{1}{t}\int_0^t \rho^2(\tau)d\tau \quad \text{and} \quad b(t) = \frac{1}{t}\int_0^t {\rho^*}^2(\tau)d\tau.$$

PROPOSITION 3.8. *If $\{u(t)\}$ be any curve in H satisfying the property $(*)$. Then $\lim_{t\to\infty} a(t) = 2(d^2 - \|p\|^2)$, where p is the weak limit of $\sigma(t)$.*

Proof. By Corollary 3.7, there exists $p \in H$ such that $\sigma(t) \rightharpoonup p$ as $t \to \infty$. Hence we have

$$\begin{aligned}\frac{1}{t}\int_0^t \rho^2(\tau)d\tau &= \frac{1}{t}\int_0^t (\lim_{s\to\infty} \|u(\tau) - u(s)\|^2)d\tau \\ &= \frac{1}{t}\int_0^t \|u(\tau)\|^2 d\tau + d^2 - 2(\lim_{s\to\infty} u(s), \sigma(t)) \\ &\xrightarrow[t\to\infty]{} 2(d^2 - \lim_{s\to\infty}(u(s), p)) = 2(d^2 - \|p\|^2).\end{aligned}$$

Therefore, $\lim_{t\to\infty} a(t) = 2(d^2 - \|p\|^2)$.

Now we study the strong convergence as $t \to \infty$ of the Cesàro mean $\sigma(t)$ of AONEC $\{u(t)\}$ in H.

PROPOSITION 3.9. *Let $\{u(t)\}$ be an curve in H satisfying the property $(*)$. Then $\sigma(t)$ converges strongly as $t \to \infty$ to the asymptotic center of the curve $\{u(t)\}$.*

Proof. We already know by Corollary 3.7 that $\sigma(t) \rightharpoonup p$ as $t \to \infty$, where p is the asymptotic center of $\{u(t)\}$. Therefore to strong convergence it is enough to show that $\|\sigma(t)\|^2 \to \|p\|^2$. Now we have

$$\begin{aligned}\|\sigma(t)\|^2 &= \frac{1}{t^2}\int_0^t\int_0^t (u(r), u(s))dsdr \\ &= \frac{1}{2t^2}\int_0^t\int_0^t (\|u(r)\|^2 + \|u(s)\|^2 - \|u(r) - u(s)\|^2)dsdr \\ &= \frac{1}{t}\int_0^t \|u(r)\|^2 dr - \frac{1}{2t^2}\int_0^t\int_0^t \rho^2(r)dsdr - c(t),\end{aligned}$$

where

$$c(t) = \frac{1}{2t^2}\int_0^t\int_0^t (\|u(r) - u(s)\|^2 - \rho^2(r))dsdr \geq 0, \text{ for } t > 0.$$

Now let

$$d(t) = \frac{1}{2t^2}\int_0^t \int_0^t (\|u(r)+u(s)\|^2 - {\rho^*}^2(r))dsdr \geq 0, \text{ for } t > 0.$$

Since $\rho^2(r) + {\rho^*}^2(r) = 2\|u(r)\|^2 + 2d^2$, for all $r \geq 0$, we have,

$$\begin{aligned} & c(t) + d(t) \\ &= \frac{1}{2t^2}\int_0^t \int_0^t (2(\|u(r)\|^2 + \|u(s)\|^2) - (\rho^2(r) + {\rho^*}^2(r)))dsdr \\ &= \frac{2}{t}\int_0^t \|u(s)\|^2 ds - 2d^2 \xrightarrow[t\to\infty]{} 0. \end{aligned}$$

Therefore

$$0 \leq \lim_{t\to\infty} c(t) \leq \lim_{t\to\infty}(c(t)+d(t)) = 0.$$

Hence $\lim_{t\to\infty} c(t) = 0$. On the other hand, by Proposition 3.8,

$$\begin{aligned} \frac{1}{2t^2}\int_0^t \int_0^t \rho^2(r)dsdr &= \frac{1}{2t}\int_0^t \rho^2(\tau)d\tau \\ &\xrightarrow[t\to\infty]{} d^2 - \|p\|^2. \end{aligned}$$

and hence

$$\begin{aligned} \|\sigma(t)\|^2 &= \frac{1}{t}\int_0^t \|u(r)\|^2 dr - \frac{1}{2t^2}\int_0^t \int_0^t \rho^2(r)dsdr - c(t) \\ &\xrightarrow[t\to\infty]{} d^2 - (d^2 - \|p\|^2) = \|p\|^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|\sigma(t) - p\|^2 &= \|\sigma(t)\|^2 - 2(\sigma(t), p) + \|p\|^2 \\ &\xrightarrow[t\to\infty]{} \|p\|^2 - 2\|p\|^2 + \|p\|^2 = 0. \end{aligned}$$

This completes the proof.

REMARK 3.12. (1) Our proof of Proposition 3.9 is of interest in view of use of almost odd nonexpansive curve. Compare this with Theorem 3.11 in [8].

(2) For the asymptotic properties of almost odd nonexpansive sequences, see [9].

4. Asymptotic behavior of solutions in H

In this section we give a main result concerning the asymptotic behavior as $t \to \infty$ of the Cesàro mean of generalized solutions of (E). Following Aizicovici [1], we assume the following conditions:

(C1) G satisfies (1.1), (2.1) and (2.2).

(C2) For every $u,\ v \in C([0,\infty) : \overline{D(A)})$,

$$\int_s^t (G(u)(\tau) - G(v)(\tau), u(\tau) - v(\tau))d\tau \geq 0, \quad 0 \leq s \leq t < \infty.$$

(C3) $G(v) \in L^1([0,\infty) : H)$, for each constant function $v(t) \equiv v \in D(A)$.

(C4) $f \in L^1([0,\infty) : H)$.

(C5) $x \in \overline{D(A)}$.

We begin with a simple lemma which will play a crucial role in our results. For proof, see [1, 12].

LEMMA 4.1. *Let A be a maximal monotone operator on H. Assume that* (C1) *hold. Let $f,\ \hat{f} \in L^1_{loc}([0,\infty) : H)$ and $x,\ \hat{x} \in \overline{D(A)}$. Let $u,\ \hat{u}$ be the corresponding generalized solutions of* (E). *If* (C2) *satisfied, then*

$$\|u(t) - \hat{u}(t)\| \leq \|u(s) - \hat{u}(s)\| + \int_s^t \|f(\tau) - \hat{f}(\tau)\|d\tau \tag{4.1}$$

for $0 \leq s \leq t < \infty$.

COROLLARY 4.2. *Let A be a maximal monotone operator on H, and $f \in L^1_{loc}([0,\infty) : H)$. Assume that* (C1), (C2), (C3) *and* (C5) *hold. If u is a generalized solution of* (E), *then for any $r,\ s,\ h \geq 0$,*

$$\|u(r+h)-u(s+h)\| \leq \|u(r)-u(s)\| + \int_s^{s+h} \|f(\tau+(r-s))-f(\tau)\|d\tau.$$

Proof. It is enough to apply Lemma 4.1 with $\hat{f}(t) = f(t+(r-s))$ and $\hat{u}(t) = u(t+(r-s))$.

PROPOSITION 4.3. *Let A be a maximal monotone operator on H. Assume that* (C1), (C2), (C3), (C4) *and* (C5) *hold. If u is a generalized solution of* (E) *and if $\{u(t)\}$ is bounded on $[0,\infty)$, then the curve $\{u(t)\}$ is an ANEC in H.*

Proof. By (C4), we have

$$\lim_{\substack{s,r\to\infty \\ r>s}} \int_s^{\infty} \|f(\tau+(r-s))-f(\tau)\|d\tau = 0.$$

Thus the result follows from Corollary 4.2 and Remark 3.3 (b) by taking

$$\epsilon_1(r,s) = \begin{cases} \displaystyle\int_s^{\infty} \|f(\tau+(r-s))-f(\tau)\|d\tau & \text{if } r \geq s \\ \displaystyle\int_r^{\infty} \|f(\tau+(s-r))-f(\tau)\|d\tau & \text{if } s \geq r. \end{cases}$$

LEMMA 4.4 [1, 12]. *Let A be a maximal monotone operator on H. Assume that* (C1), (C2), (C3), (C4) *and* (C5) *hold. Let u be a generalized solution of* (E). *Then $\{u(t)\}$ is bounded on $[0,\infty)$ if and only if $A^{-1}0$ is nonempty. Further, we have $A^{-1}0 \subset E(u(t))$.*

Now we give the strong convergence as $t \to \infty$ of the Cesàro mean $\sigma(t)$ of solution $u(t)$ of (E).

THEOREM 4.5. *Let A be an odd maximal monotone operator on H and let* (C1), (C2), (C3), (C4) *and* (C5) *be satisfied. Assume further that G is an odd mapping, i.e., $G(-u)(t) = -G(u)(t)$ for $t \geq 0$. Let u be a generalized solution of* (E) *and $\sigma(t) = \frac{1}{t}\int_0^t u(\tau)d\tau$. Then $\sigma(t)$ converges strongly as $t \to \infty$ to the asymptotic center of the curve $\{u(t)\}$.*

Proof. First, we note that since A is an odd maximal monotone operator, we have $0 \in A^{-1}0$ so that $\{u(t)\}$ is bounded on $[0, \infty)$ and $\lim_{t\to\infty} \|u(t)\|$ exists by Lemma 4.4. Hence $\{u(t)\}$ is an ANEC by Proposition 4.3. Now, let us prove that $\{u(t)\}$ is an AONEC. To this end, first we show that for the strong solutions u and $\hat{u}$ of (E) with f and $\hat{f}$, respectively,

$$\|u(t) + \hat{u}(t)\| \leq \|u(s) + \hat{u}(s)\| + \int_s^t \|f(\tau) + \hat{f}(\tau)\| d\tau. \tag{4.2}$$

for $0 \leq s \leq t < \infty$. In fact, since A is odd and monotone, it follows that

$$\begin{aligned} 0 &\leq (f(t) - u'(t) - G(u)(t) + \hat{f}(t) - \hat{u}'(t) - G(\hat{u})(t), u(t) + \hat{u}(t)) \\ &= (f(t) + \hat{f}(t), u(t) + \hat{u}(t)) - (u'(t) + \hat{u}'(t), u(t) + \hat{u}(t)) \\ &\qquad - (G(u)(t) + G(\hat{u})(t), u(t) + \hat{u}(t)) \\ &\leq \|f(t) + \hat{f}(t)\| \|u(t) + \hat{u}(t)\| - \frac{1}{2}\frac{d}{dt}\|u(t) + \hat{u}(t)\|^2 \\ &\qquad - (G(u)(t) + G(\hat{u})(t), u(t) + \hat{u}(t)) \end{aligned}$$

for $t > 0$. Integrating on $[s, t]$, from the oddness of G and (C2), we have

$$\begin{aligned} &\frac{1}{2}\|u(t) + \hat{u}(t)\|^2 \\ &\leq \frac{1}{2}\|u(s) + \hat{u}(s)\|^2 + \int_s^t \|f(\tau) + \hat{f}(\tau)\| \|u(\tau) + \hat{u}(\tau)\| d\tau. \end{aligned} \tag{4.3}$$

The inequality (4.2) follows from (4.3) and Gronwall's lemma (see [4. Lemma A.5]).

By density, the inequality (4.2) also holds for the generalized solutions u and $\hat{u}$ of (E) with f and $\hat{f}$, respectively. Then it follows with $\hat{f}(t) = f(t + (r - s))$ and $\hat{u}(t) = u(t + (r - s))$ that

$$\|u(r+h)+u(s+h)\| \leq \|u(r)+u(s)\| + \int_s^{s+h} \|f(\tau+(r-s))+f(\tau)\|d\tau.$$

So, we have

$$\|u(r+h) + u(s+h)\| \leq \|u(r) + u(s)\| + \epsilon_1(r,s), \tag{4.4}$$

where

$$\epsilon_1(r,s) = \begin{cases} \displaystyle\int_s^{\infty} \|f(\tau + (r-s)) + f(\tau)\|d\tau & \text{if } r \geq s \\ \displaystyle\int_r^{\infty} \|f(\tau + (s-r)) + f(\tau)\|d\tau & \text{if } s \geq r. \end{cases}$$

Since it also follows from (C4) that

$$\lim_{\substack{s,r\to\infty \\ r>s}} \int_s^{\infty} \|f(\tau + (r-s)) + f(\tau)\|d\tau = 0,$$

we have $\lim_{s,r\to\infty} \epsilon_1(r,s)$. Thus, from (4.4) we conclude that for any $r,\ s,\ h \geq 0$,

$$\|u(r+h) + u(s+h)\|^2 \leq \|u(r) + u(s)\|^2 + \epsilon(r,s),$$

where $\epsilon(r,s) = (2\|u(r) + u(s)\| + \epsilon_1(r,s))\epsilon_1(r,s)$,

$$\lim_{r,s\to\infty} \epsilon(r,s) \leq \lim_{r,s\to\infty} (2M + \epsilon_1(r,s))\epsilon_1(r,s) = 0,$$

and $M = \sup_{t\in[0,\infty)} \|u(t)\|$. That is, $\{u(t)\}$ is an AONEC in H. Therefore the result follows from Proposition 3.9.

REMARK 4.6. (1) The case in which $G \equiv 0$ was previously considered by Djafari Rouhani [8].

(2) As in [2, 6, 11, 14, 15], our results can be used to study the asymptotic behavior of solutions of the related nonlinear Volterra equation:

$$u(t) + \int_0^t b(t-s)Au(s)ds \ni g(t), \quad t > 0$$
$$u(0) = x = g(0).$$

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