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STUDY OF THE BEST DEFORMATION FOR EXTENDING HARDY SPACES AND ITS APPLICATIONS

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Let D be an open unit disc, T be the unit circle in the complex plane. The Hardy space H^p $(0 < p < \infty)$ consists of all functions holomorphic in D for which

$$||f||_p = \begin{cases} \lim_{r\to 1}[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta]^{\frac{1}{p}}, & 0 < p < \infty \\ \sup_{z\in D} |f(z)|, & \mathrm{p} = \infty \end{cases}$$

is finite. If we define

$$M_p(f,r) = \{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\}^{\frac{1}{p}}, \quad 0 < p < \infty$$
$$M_\infty(f,r) = \sup\{|f(re^{i\theta})| : 0 \le \theta \le 2\pi\},$$

we can rewrite

$$||f||_p = \lim_{r\to 1} M_p(f,r), \quad 0 < p \le \infty.$$

A function bounded and holomorphic in D is said to be an inner function if its boundary values have modulus 1 almost everywhere. A Blaschke sequence is a (finite or infinite) sequence $\{a_n\}$ of complex numbers satisfying the conditions; $0 < |a_n| < 1$ and $\sum(1 - |a_n|)$ is finite. An important class of inner function is the Blaschke product. A Blaschke product $B(z)$ with zeros $\{a_n\}$ is a function defined by a formula;

$$B(z) = \prod \frac{|a_n|}{a_n}\frac{a_n - z}{1 - \bar{a}_n z}$$

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for a Blaschke sequence $\{a_n\}$. The set of Blaschke products is uniformly dense in the set of inner function by the Frostman's theorem [9,10].

Let B^p $(0 < p < 1)$ denote the space of functions $f(z)$ holomorphic in D for which

$$\frac{1}{2\pi}\int_0^1\int_0^{2\pi}|f(re^{i\theta})|(1-r)^{\frac{1}{p}-2}d\theta dr$$

is finite. If we use the quantity $M_p(f,r)$, we can rewrite as following;

$$||f||_{B^p} = \int_0^1 (1-r)^{\frac{1}{p}-2}M_1(f,r)dr.$$

It turns out H^p is a subspace of B^p, especially $B^p = H^p$ for $p = \frac{1}{2}$. It was found [3,12], H^p is dense in B^p and two spaces have the same continuous linear functionals. This makes it possible to identify B^p with the closure of H^p in its second dual of H^p [1,5]. Thus this deformation space B^p is in some respects nicer than H^p space.

Now we introduce the weighted Bergman space $A^{p,q}$ [7,11] and develop some of properties on $A^{p,q}$ space. If $f(z)$ is holomorphic in D and $0 < p < 1$ and $q > 0$ we define the weighted L^q norm by

$$\int_0^1\int_0^{2\pi}|f(re^{i\theta})|^q(1-r)^{\frac{1}{p}-2}d\theta dr.$$

If it is finite we say $f(z)$ belongs to $A^{p,q}$. This space is more extended than B^p, especially $A^{p,q}$ is equal to B^p when $q = 1$, at that time there are many interesing results on it.

G. Caughram and L. Shields raised the question whether there exits a singular inner function whose derivative is in H^p $(p = \frac{1}{2})$. L. Duren, W. Romberg and L. Shields [6] proved that the derivative of every Blaschke product lies in B^p for all p $(0 < p < \frac{1}{2})$. In [16], W. Rudin showed that if the zeros $\{a_n\}$ of a Blaschke product satisfying the condition

$$\sum(1-|a_n|)\log\frac{1}{1-|a_n|} < \infty,$$

then the derivative of $B(z)$ lies in $B^{\frac{1}{2}}$. Also, P. Ahern and D. Clark [2] proved that if $\sum(1-|a_n|)^{\frac{1}{2}} < \infty$, and $\sum(1-|a_n|)^{\frac{1}{2}}\log\frac{1}{1-|a_n|} =$

∞, then there is a Blaschke product $B(z)$ with zerors $\{a_n\}$ satisfying $B'(z) \in B^{\frac{2}{3}}$ and $B'(z) \in H^{\frac{1}{2}}$. D. Protas [15] generalized this property as followings that if the zerors $\{a_n\}$ satisfy $\sum(1-|a_n|)^\alpha < \infty$ then $B'(z) \in B^{\frac{1}{1+\alpha}}$ for some $\alpha \quad (0<\alpha<1)$, and $B'(z) \in H^{1-\alpha}$ for $\alpha \quad (0 < \alpha < \frac{1}{2})$. We could not translate all the B^p results into $A^{p,q}$ space.

In this paper, we find some results in deformations of H^p spaces and consider the relation between the distribution of $B(z)$ and $\hat{B}(z)$. There are several known conditions on the distribution of Blaschke sequences that imply the derivative of Blaschke products lies in the extended H^p space. The basic problem we consider is that of determining $A^{p,q}$ spaces to which the derivative of $B(z)$ belongs.

For typographical reasons we frequently omit the superscript p in writing $||f||_{B^p}$. We first prove followings.

PROPOSITION 1. *For each f in B^p, the following inequality holds for constant K_p (: depend on p).*

$$|f(z)| \leq K_p ||f||_B (1-r)^{-\frac{1}{p}}.$$

Proof. Let $R < r < 1$, then

$$\begin{aligned} ||f||_B &\geq \int_R^1 (1-r)^{\frac{1}{p}-2} M_1(f,r) dr \\ &\geq M_1(f,R)(\frac{1}{p}-1)^{-1}(1-R)^{\frac{1}{p}-1}. \end{aligned}$$

Hence

$$M_1(f,R) \leq (\frac{1}{p}-1)||f||_B(1-R)^{1-\frac{1}{p}}.$$

From this, the estimate follows by writing

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where $R = \frac{1}{2}(1+|z|)$.

LEMMA 2. *For each $f \in B^p$, $f_\rho \to f$ in B^p-norm as $\rho \to 1$, where $f_\rho(z) = f(\rho z)$.*

Proof. Given $f \in B^p$ and $\varepsilon > 0$, choose $r < 1$ such that

$$\int_R^1 (1-r)^{\frac{1}{p}-2} M_1(f,r)dr \leq \varepsilon \quad \cdots (2.1).$$

Since $M_1(f,r)$ is an increasing function of r, (2.1) remains valid when f is replaced by f_ρ. Now choose ρ so close to 1 that $|f_\rho(z) - f(z)| < \varepsilon$ on $|z| \leq R$. Then

$$\int_0^R (1-r)^{\frac{1}{p}-2} M_1(f_\rho - f, r)dr < \varepsilon ||1||_B.$$

Combining this with (2.1), we have

$$||f_\rho - f||_B \leq \varepsilon ||1||_B + 2\varepsilon,$$

so $f_\rho \to f$ in norm as $\rho \to 1$.

LEMMA 3. *For each $f \in H^p$, $\quad ||f||_B \leq K_p ||f||_p$.*

Proof. The above statement means that $H^p \subset B^p$, and gives the norm inequality. Also, H^p contains all functions holomorphic in a bigger disc, and such functions are dense in B^p by Lemma 2.

If we use above statements, the following fact is satisfied.

THEOREM 4. *Let φ be in the dual $(B^p)^*$ of B^p for $0 < p < 1$, then there is unique function g such that*

$$\varphi(f) = \lim_{r \to 1} \int_0^{2\pi} f(re^{i\theta}) g(e^{-i\theta}) d\theta, \quad f \in B^p,$$

where $g(z)$ is holomorphic in D and continuous on $\bar{D}$.

We consider some relations between the distribution of the zeros of the k-th derivative $B^{(k)}(z)$ of Blaschke product and the behavior of its Taylor coefficients

$$\hat{B}(z) = \frac{B^{(k)}(0)}{k!} \quad (k \geq 0).$$

Let $f(x)$ be defined in a closed interval I and let

$$\omega(\delta) = \omega(\delta, f) = \sup |f(x_2) - f(x_1)|$$

for $x_1, x_2 \in I, \quad |x_2 - x_1| \leq \delta$. The function $\omega(\delta)$ is called the modulus of continuity of f. If I is finite, then f is continuous in I if and only if $\omega(\delta) \to 0 \quad$ as $\quad \delta \to 0$. For some $\alpha > 0$, we have $\omega(\delta) \leq c\delta^{\alpha}$, where c is independent of δ.

Recall that $f(z)$ satisfies a Lipschitz condition of order n in D (denote $f \in \Lambda_n$) if and only if

$$|f(z_1) - f(z_2)| \leq c|z_1 - z_2|^n$$

for $0 < n \leq 1$ where c is independent of z_1, z_2 and $z_1, z_2 \in D$. Similarly, $f \in \Lambda_n^*$ means that

$$|f(z_1) - f(z_2)| = o(|z_1 - z_2|^n).$$

It is obvious that functions in classes Λ_n, Λ_n^* are bounded and continuous. Only the case $\quad 0 < n < 1 \quad$ is interesting: if $\quad n > 1$, then $\omega(\delta)/\delta \quad$ tends to zero with $\delta, \quad f'(x)$ exists and is zero everywhere, and f is a constant. The function f belongs to Λ_1 if and only if f is integral of a bounded function.

Here, we apply these properties to the Hardy space. We recall that the function $f \in L^p(T)$ is in the class $\Lambda_\alpha^p, \quad (0 < \alpha < 1, 1 \leq p < \infty)$ if its L^p-modulus of continuity

$$\omega_p(\delta) = \sup_{|\theta| \leq \delta} [\int_T |f(\zeta e^{i\theta}) - f(\zeta)|^p d\zeta]^{\frac{1}{p}}$$

satisfies the condition $\omega_p(\delta) \leq c\delta^\alpha$. According to the Hardy and Littlewood theorem, the boundary values of a function ψ from the Hardy space H^p belong to the class Λ_α^p if and only if

$$[\int_T |\psi'(r\zeta)|^p |d\zeta|]^{\frac{1}{p}} \leq c(1-r)^{\alpha-1}.$$

Newman and Shapiro [14] have proved that the Taylor coefficient of an inner function may have order $o(\frac{1}{k})$ only in the trivial case of finite Blaschke products. For all $B(z)$, whose zeros satisfy the Newman condition

$$\sup_{k\geq 0} \frac{(1-|a_{k+1}|)}{(1-|a_k|)} < 1,$$

J. Newman and S. Shapiro obtained the estimate

$$\hat{B}(k) = O(\frac{1}{k}),$$

where $\hat{B}(k)$ is the Taylor coefficients of $B(k)$.

THEOREM 5. *Let $B(z)$ be a Blaschke product and let $\{z_k\}$ be its zero, then the following statements are equivalent:*

(1) *the sequence $\{z_k\}$ satisfies the Newman condition,*
(2) $\hat{B}(k) = O(\frac{1}{k})$,
(3) $\sum_{k\geq n} |\hat{B}(k)|^2 = O(\frac{1}{n})$,
(4) $B(z) \in \Lambda^p_{\frac{1}{p}}$ *for some* $1 < p < \infty$, *and*
(5) $\int_T |B''(r\zeta)||d\zeta| \leq c(1-r)^{-1}$ *for some constant* c.

We show that condition (4) for $p = 2$ is equivalent to (3) (see [19]). In addition, from (5) it follows that $B(z) \in \Lambda^p_{\frac{1}{p}}$ for all $p \in (0,\infty)$ [13].

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is obvious, (5) $\Rightarrow$ (2) follows from the estimate

$$r^{k-2}k(k-1)|\hat{B}(k)| \leq \frac{1}{2\pi}\int_T |B''(r\zeta)||d\zeta|$$

for $r = 1 - \frac{1}{k}$. We show that (1) $\Rightarrow$ (5) and (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (5). We make use of the easily proved estimate [4]

$$|B''(z)| \leq 2\sum \frac{1-|z_j|^2}{|1-\bar{z}_j z|^3} + (\sum \frac{1-|z_j|^2}{|1-\bar{z}_j z|^2})^2.$$

Let $z = r\zeta$ and integrate with respect to $\zeta \in T$, then we obtain

$$\begin{aligned}\int_T |B''(r\zeta)||d\zeta| \leq & 2\sum(1-|z_j|^2)\int_T |1-\bar{z}_j r\zeta|^{-3}|d\zeta| \\ & + \{\sum(1-|z_j|^2)(\int_T |1-\bar{z}_j r\zeta|^{-4}|d\zeta|)^{\frac{1}{2}}\}^2.\end{aligned}$$

Since

$$\int_T |1-\bar{z}_j r\zeta|^{-n}|d\zeta| \leq c(1-r|z_j|)^{1-n}, \quad (n>1),$$

we have

$$\int_T |B''(r\zeta)||d\zeta| \leq c\sum \frac{1-|z_j|^2}{(1-r|z_j|)^2} + c\{\sum \frac{1-|z_j|^2}{(1-r|z_j|)^{\frac{3}{2}}}\}^2.$$

From the condition $\hat{B}(k) = O(\frac{1}{k})$ it follows [18] that for $\alpha > 1$ one has

$$\sum(1-|z_j|^2)(1-r|z_j|)^{-\alpha} \leq c(1-r)^{1-\alpha}.$$

Applying this inequality for $\alpha = 2$ and $\alpha = \frac{3}{2}$, we obtain

$$\int_T |B''(r\zeta)||d\zeta| \leq c(1-r)^{-1}$$

for some constant c.

(4) $\Rightarrow$ (1). Let $B(z) \in \Lambda^p_\alpha, \quad \alpha = \frac{1}{p}, \quad p \in (1,\infty)$. From the known results regarding the approximation by Abel means there follows [8] that

$$[\int_T |B(\zeta) - B(r\zeta)|^p |d\zeta|]^{\frac{1}{p}} \leq c(1-r)^\alpha,$$

whence

$$\int_T (1-|B(r\zeta)|)^p |d\zeta| \leq c(1-r)^{\alpha p}.$$

By using the Carleson measure, the proof is complete.

Now we apply the derivative of $B(z)$ to the deformation of B^p and find the condition that derivative of Blaschke product belongs to $A^{p,q}$ spaces. Of course we restrict the value of p within $0 < p < 1$.

THEOREM 6. *Let $B(z)$ be a Blaschke product with zerors $\{a_n\}$ such that $\sum(1-|a_n|)^q$ is finite for some $q \quad (0 < q < 1)$. Then the condition of $p \quad (0 < p < \frac{1}{2q})$ implies $B'(z) \in A^{p,q}$.*

In order to prove this theorem we use the following lemma.

LEMMA 7[17]. *Let $\{a_n\}$ be a sequence in D. Then there exists constants K, K_p(: depend on p) such that*

$$\int_0^{2\pi} \frac{1}{|1-a_n re^{i\theta}|^{2p}} d\theta \leq \begin{cases} \dfrac{K_p}{(1-|a_n|r)^{2p-1}} & \text{if } \ p > \dfrac{1}{2} \\ K & \text{if } \ p < \dfrac{1}{2}. \end{cases}$$

Proof of Theorem 6. The derivative of $B(z)$ is following formula;

$$B'(z) = \sum \frac{B_n(z)(1-|a_n|^2)}{(1-\bar{a}_n z)^2},$$

where $B_n(z) = \frac{B(z)(1-\bar{a}_n z)}{z-a_n}$, and this implies that

$$\begin{aligned} |B'(z)| &\leq \sum \frac{(1-|a_n|^2)}{|1-\bar{a}_n z|^2} \\ &\leq 2 \sum \frac{(1-|a_n|)}{|1-\bar{a}_n z|^2}. \end{aligned}$$

By the hypothesis, for fixed $q \quad (0 < q < 1)$,

$$|B'(z)|^q \leq 2^q \sum \frac{(1-|a_n|)^q}{|1-\bar{a}_n z|^{2q}}.$$

Integrate each side and use Lemma 7 for each $q \quad (\frac{1}{2} < q < 1)$, then we obtain that

$$\begin{aligned} &\int_0^1 \int_0^{2\pi} |B'(re^{i\theta})|^q (1-r)^{\frac{1}{p}-2} d\theta dr \\ &\leq 2^q K_p \sum (1-|a_n|)^q \int_0^1 (1-r)^{\frac{1}{p}-1-2q} dr \end{aligned}$$

is finite for $0 < p < \frac{1}{2q}$. If $0 < q < \frac{1}{2}$, we get that

$$\int_0^{2\pi} |B'(re^{i\theta})|^q d\theta \leq 2^q K \sum (1-|a_n|)^q.$$

Thus the proof is complete.

We are now prepared to discuss another conditions to find values of p and q or relations of its in $A^{p,q}$ space using the basic estimate of the inequality $|1-\bar{a}_n re^{i\theta}| \geq (1-r)$.

THEOREM 8. *Let $\{a_n\}$ be a Blaschke sequence with $\sum(1-|a_n|)<\infty$ and $q<\frac{1}{2p}$, then $B'(z)\in A^{p,q}$ for each $q>1$.*

Proof. We consider the estimate derived from the finite Blaschke product as it is difficult to ensure the convergence of $\sum\frac{1-|a_n|}{|1-\bar{a}_n re^{i\theta}|^{2q}}$ for given q. Let

$$B_m(z)=\prod_{n=1}^{m}\frac{|a_n|}{a_n}\frac{a_n-z}{1-\bar{a}_n z}$$

be a finite Blaschke product, then the derivative of $B_m(z)$ is following;

$$B'_m(z)=\sum_{n=1}^{m}B_n(z)\frac{1-|a_n|^2}{(1-\bar{a}_n z)^2},$$

where $B_n(z)=B_m(z)\frac{1-\bar{a}_n z}{z-a_n}$. This implies that

$$|B'_m(re^{i\theta})|^q\le 2^q(\sum_{n=1}^{m}\frac{d_n}{|1-\bar{a}_n re^{i\theta}|^2})^q$$

for $1-|a_n|=d_n \quad (n=1,2,3,\cdots)$. By the Hölder inequality, we have

$$\begin{aligned}|B'_m(re^{i\theta})|^q&\le 2^q(\sum_{n=1}^{m}(d_n^{\frac{1}{q'}})^{q'})^{\frac{q}{q'}}\sum_{n=1}^{m}\frac{d_n}{|1-\bar{a}_n re^{i\theta}|^{2q}}\\&=2^q(\sum_{n=1}^{m}d_n)^{\frac{q}{q'}}\sum_{n=1}^{m}\frac{d_n}{|1-\bar{a}_n re^{i\theta}|^{2q}}\end{aligned}$$

where $\frac{1}{q}+\frac{1}{q'}=1$. By Lemma 7,

$$\begin{aligned}\int_0^{2\pi}|B'_m(re^{i\theta})|^q d\theta&\le 2^q(\sum_{n=1}^{m}d_n)^{\frac{q}{q'}}\sum_{n=1}^{m}d_n\int_0^{2\pi}\frac{d\theta}{|1-\bar{a}_n re^{i\theta}|^{2q}}\\&\le 2^q K_q(\sum_{n=1}^{m}d_n)^{\frac{q}{q'}+1}(1-r)^{-2q+1}.\end{aligned}$$

Since $\sum d_n$ is finite, the value of the right side of the preceding inequality is finite independently of the choice m. Therefore, we have the following by the Lebesgue's theorem,

$$\int_0^{2\pi} |B'(re^{i\theta})|^q d\theta = \lim_{m\to\infty} \int_0^{2\pi} |B'_m(re^{i\theta})|^q d\theta$$
$$\leq 2^q K_q(\sum d_n)^{\frac{q}{q'}+1}(1-r)^{-2q+1}.$$

Thus

$$\int_0^1 \int_0^{2\pi} |B'(re^{i\theta})|^q (1-r)^{\frac{1}{p}-2} d\theta dr$$
$$= \lim_{m\to\infty} \int_0^1 \int_0^{2\pi} |B'_m(re^{i\theta})|^q (1-r)^{\frac{1}{p}-2} d\theta dr$$
$$\leq 2^q K_q(\sum d_n)^{\frac{q}{q'}+1} \int_0^1 (1-r)^{\frac{1}{p}-2q-1} dr.$$

By the hypothesis, this integration is finite for $q < \frac{1}{2p}$. Therefore the proof is complete.

We notice the convergent relation of $\sum(1-|a_n|)^q$ and $\sum(1-|a_n|)$ is depend on q in the proof of the above theorem.

REMARK. *If* $\sum(1-|a_n|)^q$ *is finite then* $\sum(1-|a_n|)$ *is also finite but the converse does not hold for each* $q < 1$. *On the other hand, this property is opposite to the mentioned argument for each* $q > 1$.

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