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THE DECOMPOSITIONS OF Γ–NEAR–RINGS

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In this paper, all near-rings considered will be right near-rings. We refer to Pilz [4] for all notations and conventions. Berman and Silverman [1] showed that any near-ring N can be written as the sum of the two subnear-rings $C(N)$ and $Z(N)$. Now we consider the decompositions of Γ-near-rings.

DEFINITION 1. *A Γ-near-ring is a triple* $(M, +, \Gamma)$, *where*

(1) $(M, +)$ *is a group,*

(2) Γ *is a nonempty set of binary operators such that* $(M, +, \gamma)$ *is a near-ring for each* $\gamma \in \Gamma$,

(3) $x\gamma(y\mu z) = (x\gamma y)\mu z$ *for all* $x, y, z \in M, \gamma, \mu \in \Gamma$.

Let M be a Γ-near-ring. If a subgroup A of $(M, +)$ is a subnear-ring of $(M, +, \gamma)$ for each $\gamma \in \Gamma$, then we say A is a sub-Γ-near-ring of M. A normal subgroup A of $(M, +)$ such that $x\gamma(a + y) - x\gamma y \in A$ and $a\gamma x \in A$ for all $a \in A, x, y \in M$ and $\gamma \in \Gamma$, is called an ideal of M. The zerosymmetric part of M, M_0 is the set $\{x \in M : x\gamma 0 = 0 \quad for \quad all \quad \gamma \in \Gamma\}$. The constant part of M, M_c is the set $\{x \in M : x\gamma 0 = x \quad for \quad all \quad \gamma \in \Gamma\}$. We note that if $x\gamma 0 = x$ for some $\gamma \in \Gamma$, and $\mu \in \Gamma$, then $x\mu 0 = (x\gamma 0)\mu 0 = x\gamma(0\mu 0) = x\gamma 0 = x$. Hence, $x \in M_c$ if and only if there exists $\gamma \in \Gamma$ such that $x\gamma 0 = x$. M is said to be zerosymmetric if $M = M_0$. Throughout this paper, M denotes a zerosymmetric Γ-near-ring. An idempotent is an element $x \in M$ such that $x\gamma x = x$ for all $\gamma \in \Gamma$.

THEOREM 2. *Let e be an idempotent in M. Then every element $x \in M$ can be expressed as two sums $x = x\gamma e + (-x\gamma e + x) = (x - x\gamma e) + x\gamma e$ for each $\gamma \in \Gamma$ and $M = A \oplus B = B \oplus A$, where $A = \{x\gamma e | x \in M\}$ and $B = \{x \in M | x\gamma e = 0\}$.*

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Proof. Clearly A and B are sub-Γ-near-rings of M. Also the elements $-x\gamma e + x$ and $x - x\gamma e$ are in B. suppose that $x = a_1 + b_1 = a_2 + b_2, a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then $-a_2 + a_1 = b_2 - b_1$ must be in $A \cap B$. But the only element in $A \cap B$ is 0. For if $a \in A \cap B$, then $a\gamma e = 0$ and $a = x\gamma e$ for some $x \in M$. So we have $0 = a\gamma e = (x\gamma e)\gamma e = x\gamma(e\gamma e) = x\gamma e = a$. Thus $a_1 = a_2$ and $b_1 = b_2$. The uniqueness of the other representation is proved in the same way. Therefore we have $M = A \bigoplus B = B \bigoplus A$.

Since 0 is an idempotent in M, we have the following:

COROLLARY 3. *For each $\gamma \in \Gamma$, we get that $M = M_c^\gamma \bigoplus M_0^\gamma$, where $M_c^\gamma = \{x \in M | x\gamma 0 = x\}$ and $M_0^\gamma = \{x \in M | x\gamma 0 = 0\}$.*

It is also easy to see that M_0^γ is a left ideal.

DEFINITION 4. *A Γ-near-ring M is transitive if for each $x_1, x_2 \in M_c^\gamma, x_1 \neq 0$, there exists $y \in M_0^\gamma$ such that $y\gamma x_1 = x_2$.*

THEOREM 5. *Let M be a transitive $\Gamma-$near-ring. Then M_0^γ is a maximal sub-Γ-near-ring of M for each $\gamma \in \Gamma$.*

Proof. Let X be a sub-Γ-near-ring of M with $M_0^\gamma \subsetneqq X$. For $x \in X, x \notin M_0^\gamma$, we have $x - x\gamma 0 \in M_0^\gamma$ and hence $x\gamma 0 \in X$. But $x\gamma 0 \in M_c^\gamma$ and $x\gamma 0 \neq 0$. Since M is transitive, $y\gamma(x\gamma 0) = 0$ for some $y \in M_0^\gamma$ and hence $M_c^\gamma \subseteq X$. Thus $X = M$. Therefore M_0^γ is a maximal sub-Γ-near-ring of M.

From now on, we consider the transitive Γ-near-ring M.

THEOREM 6. *If X is a subgroup of $(M, +)$, $M_0^\gamma \subset X$, and $m_1\gamma(m_2 + x) - m_1\gamma m_2 \in X$ for $m_1, m_2 \in M, x \in X$ then $M = X$. Hence M_0^γ is a maximal left ideal of M for each $\gamma \in \Gamma$.*

Proof. Note that $M_0^\gamma \gamma X \subseteq X$ for each $\gamma \in \Gamma$. Applying the method used in the proof of the above Theorem 5, we have our results.

Let M be a Γ-near-ring and A a right ideal of M. Then $M_0^\gamma \gamma A \subseteq A$ for each $\gamma \in \Gamma$. We say M is simple if its only ideals are (0) and M.

PROPOSITION 7. *If A is an ideal of M, $M_0^\gamma \cap A \neq (0)$, and M_0^γ is a simple for each $\gamma \in \Gamma$, then $M_0^\gamma \subseteq A$.*

Proof. Note that $M_0^\gamma \cap A$ is a nonzero left ideal of M_0^γ. Also $(M_0 \cap A)\gamma M_0^\gamma \subseteq M_0^\gamma \cap A$. So $M_0 \cap A$ is an ideal of M_0^γ and since M_0^γ is simple, we have $M_0^\gamma \cap A = M_0^\gamma$, hence $M_0^\gamma \subseteq A$.

PROPOSITION 8. *If A is an ideal of M and $A \cap M_0^\gamma = (0)$ for each $\gamma \in \Gamma$, then $A = M_c^\gamma$ or $A = (0)$.*

Proof. Since A is a right ideal of M, for $x = c + z \in A$ $(c \in M_0^\gamma, z \in M_c^\gamma)$, we have $x\gamma 0 = z \in A$ for $\gamma \in \Gamma$. So if $A \cap M_0^\gamma = (0)$, then $A \subset M_c^\gamma$. For each nonzero $a \in A$ and for any $z \in M_c^\gamma$, we can write $a\gamma c = z$ for some $c \in M_0^\gamma$. But $a\gamma c \in A$, so $M_c^\gamma \subset A$. Thus we have $A = M_c^\gamma$.

References

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