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SOME FIXED POINT THEOREMS ON H–SPACES (II)

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1. Introduction and Preliminaries

The following famous Fan-Browder's fixed point theorem plays important roles in nonlinear analysis. Recently it has been generalized and extended by many authors [2-8]. In this paper we generalize the Fan-Browder's fixed point theorem using the particular form of the generalized H-KKM theorem due to Chang-Ma [3].

THEOREM 1.1 (FAN-BROWDER [2,4]). *Let X be a Hausdorff topological vector space, K a nonempty compact convex subset of X and $T:K \to 2^K$ a mapping satisfying:*

(1) *for each $x \in K$, $T(x)$ is nonempty convex,*
(2) *for each $y \in K$, $T^{-1}(y) = \{x \in K : y \in T(x)\}$ is open in K.*

Then T has a fixed point.

For the sake of conveniences we recall some definitions and notations [1] needed in section 2.

DEFINITION 1.1. *An H-space is a pair $(X, \{\Gamma_A\})$, where X is a topological space, and $\{\Gamma_A\}$ is a given family of nonempty contractible subsets of X indexed by the finite subset of X such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$*

DEFINITION 1.2. *Let $(X, \{\Gamma_A\})$ be an H-space, D be a nonempty subset of X.*

(1) *D is said to be H-convex if, for every finite subset $A \subset D$, it follows that $\Gamma_A \subset D$.*
(2) *D is said to be weakly H-convex if, for evey finite subset $A \subset D$, $\Gamma_A \cap D$ is nonempty and contractible.*

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(3) *A subset* $K \subset X$ *is said to be H-compact if , for every finite subset* $A \subset X$, *there exists a compact weakly H-convex subset* $D \subset X$ such that $K \cup A \subset D$.

DEFINITION 1.3. *In a given H-space* $(X, \{\Gamma_A\})$ *a mapping* $F : X \to 2^X$ *is called an H-KKM mapping if* $\Gamma_A \subset \bigcup_{x\in A} F(x)$ *for each finite subset* $A \subset X$.

DEFINITION 1.4. *A subset D of a topological space X is called compactly open (respectively, compactly closed) if for evey compact set* $K \subset X$, *the set* $D \cap K$ *is open (respectively, closed) in K.*

REMARK 1.5. *It is easily shown that a closed-valued mapping (respectively, open-valued)* $F{:}X \to 2^X$ *is compactly closed (respectively, compactly open). And a mapping* $F{:}X \to 2^X$ *is compactly open if and only if the mapping* $G{:}X \to 2^X$ *defined by, for every* $x \in X$, $G(x) = Y \setminus F(x)$ *is compactly closed.*

2. Main Results.

THEOREM 2.1 (CHANG-MA [3]). *Let* $(X, \{\Gamma_A\})$ *be an H-space and* $G{:}X \to 2^X$ *H-KKM mapping with compactly open (closed) values. Then every finite subfamily of* $\{G(x) : x \in X\}$ *has a nonempty intersection. Moreover, if G is compactly closed-valued and there exists an* $x_0 \in X$ *such that* $G(x_0)$ *is compact , then we have* $\bigcap_{x\in X} G(x) \neq \emptyset$.

THEOREM 2.2. *Let* $(X, \{\Gamma_A\})$ *be an H-space and* $S{:}\ X \to 2^X$ *a mapping satisfying:*

(1) $S^{-1}(y)$ is *H-convex for each* $y \in X$,

(2) $T(x) = X \setminus S(x)$ *is not an H-KKM mapping for each* $x \in X$.

Then S has a fixed point.

Proof. Since $T{:}X \to 2^X$ is not an H-KKM mapping, there exists a finite subset A of X such that $\Gamma_A \not\subset \bigcup_{x\in A} T(x)$. Hence there exists $x_0 \in \Gamma_A$ such that $x_0 \in S(x)$ for each $x \in A$, i. e. , $x \in S^{-1}(x_0)$ for each $x \in A$. Since $S^{-1}(x_0)$ is H-convex, thus $\Gamma_A \subset S^{-1}(x_0)$. Hence $x_0 \in S^{-1}(x_0)$, i. e. , S has a fixed point.

The following two theorems are our main results which generalize and extend the famous Fan-Browder's fixed point theorem to H-spaces.

THEOREM 2.3. *Let $(X, \{\Gamma_A\})$ be a compact H-space, and $T{:}X \to 2^X$ a mapping satisfying:*

(1) *$T(x) \neq \emptyset$, and H-convex for each $x \in X$*

(2) *$T^{-1}(y)$ is compactly open for each $y \in X$.*

Then T has a fixed point.

Proof. Define a mapping $G : X \to 2^X$ by $G(y) = X \setminus T^{-1}(y)$ for each $y \in X$. Then G: $X \to 2^X$ is not an H-KKM mapping. In fact, if G is an H-KKM mapping, then by Theorem 2.1, $\bigcap_{y\in X} G(y) \neq \emptyset$, i. e. , there exists a $y_0 \notin T^{-1}(y)$, i. e. , $y \notin T(y_0)$ for each $y \in X$, then $T(y_0) = \emptyset$. This contradicts to the condition (1). By Theorem 2.2, T has a fixed point.

THEOREM 2.4. *Let $(X, \{\Gamma_A\})$ be a compact H-space, and $T{:}X \to 2^X$ a mapping satisfying:*

(1) *$T(x) \neq \emptyset$, and H-convex for each $x \in X$,*

(2) *for each $y \in X$, $T^{-1}(y)$ contains a compactly open subset O_y of X,*

(3) *$\bigcup_{x\in X} O_x = X$.*

Then T has a fixed point.

Proof. If we set $G(x) = X \setminus O_x$ for each $x \in X$, then $G : X \to 2^X$ is not an H-KKM mapping. In fact, if G is an H-KKM mapping, then by Theorem 2.1, $\bigcap_{x\in X} G(x) \neq \emptyset$, i. e. , $\bigcap_{x\in X}(X \setminus O_x) \neq \emptyset$. This contradicts to the condition (3). Hence there must exist at least one finite subset A of X such that $\Gamma_A \not\subset \bigcup_{x\in A} G(x)$, i. e. , $y \in \Gamma_A$ implies $y \in X \setminus G(x)$ for each $x \in A$. Thus $y \in O_x \subset T^{-1}(x)$, i. e. , $x \in T(y)$. Since $T(y)$ is H-convex, $\Gamma_A \subset T(y)$, i. e. , $y \in T(y)$. This completes the proof.

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