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COMMON FIXED POINTS OF COMPATIBLE MAPPINGS OF TYPE (A)

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1. Introduction

Sessa [13] defined a generalization of commutativity, which is called weak commutativity. Further Jungck [3] introduced more generalized commutativity, so called compatiblity, which is more than that of weak commutativity mappings. The utility of compatiblity in the context of fixed point theory was demonstrated by extending a theorem of Park-Bae [10]. Also, Jungck [4] extended the results of Khan-Imdad [8] and Singh-Singh [14] by employing compatible mappings in lieu of commuting mappings and by using four functions as opposed to three and also gave an interesting connecting with his concept in his consecutive paper [5] Recently, Kang-Cho-Jungck [7] extended the results of Ding [1], Diviccaro-Sessa [2] and Jungck [4] by using any one continuous and employing compatible mappings.

Most recently, Jungck-Murthy-Cho [6] introduced the concept of compatible mappings of type (A) in metric spaces, which is equivalent to the concept of compatible mappings under some conditions and proved common fixed point theorems of compatible mappings of type (A) on a metric space which improve the results of Pathak [11] and Prasad [12].

In this paper, we prove the results of Ding [1], Diviccaro-Sessa [2], Jungck [4] and Kang-Cho-Jungck [7] for two pairs of compatible mappings of type (A) and also give two examples to illustrate our main theorems.

Jungck [3] defined the following:

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DEFINITION 1.1. Let A and S be mappings from a metric space (X,d) into itself. Then the mappings A and S are said to be *compatible* if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X.

The following are given by Jungck-Murthy-Cho [6].

DEFINITION 1.2. Let A and S be mappings from a metric space (X,d) into itself. Then the mappings A and S are said to be *compatible of type* (A) if $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$ and $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$, when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X.

PROPOSITION 1 3. *Let A and S be continuous mappings from a metric space (X,d) into itself. Then A and S are compatible if and only if they are compatible of type (A).*

We use the following properties of compatible mappings of type (A) for our main theorem:

PROPOSITION 1.4. *Let A and S be compatible mappings of type (A) from a metric space (X,d) into itself. If $At = St$ for some t in X, then $ASt = SSt = SAt = AAt$.*

PROPOSITION 1.5. *Let A and S be compatible mappings of type (A) from a metric space (X,d) into itself. Suppose that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X. Then $\lim_{n\to\infty} SAx_n = At$ if A is continuous at t.*

2. Common Fixed Points of Four Mappings

Throughout this paper, suppose that the function $\phi : [0,\infty)^5 \to [0,\infty)$ satisfies the following conditions:

(i) ϕ is upper-semicontinuous and non-decreasing in each coordinate variable,

(ii) $\psi(t) = \max\{\phi(0,0,t,t,t),\ \phi(t,t,t,0,2t),\ \phi(t,t,t,2t,0)\} < t$ for each $t > 0$.

Our main result is the following theorem:

THEOREM 2.1. *Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions:*

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X), \tag{2.1}$$

$$\begin{aligned} d(Ax, By) \leq \phi\big(&d(Ax, Sx), d(By, Ty), \\ &d(Sx, Ty), d(Ax, Ty), d(By, Sx)\big) \end{aligned} \tag{2.2}$$

for all x, y in X, where ϕ satisfies (i) *and* (ii). *Suppose that*
(2.3) *one of A, B, S and T is continuous,*
(2.4) *the pairs A, S and B, T are compatible mappings of type (A).*
Then A, B, S and T have a unique common fixed point in X.

Let x_0 be an arbitrary point in X. Since (2.1) holds, we can choose a point x_1 in X such that $y_1 = Tx_1 = Ax_0$, and for this point x_1, there exists a point x_2 in X such that $y_2 = Sx_2 = Bx_1$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \quad \text{and} \quad y_{2n} = Sx_{2n} = Bx_{2n-1} \tag{2.5}$$

for $n = 1, 2, \cdots$.

LEMMA 2.2 ([9]). *Suppose that $\psi : [0, \infty) \to [0, \infty)$ is non-decreasing and upper-semicontinuous. If $\psi(t) < t$ for every $t > 0$, then $\lim_{n \to \infty} \psi^n(t) = 0$, where $\psi^n(t)$ denotes the composition of $\psi(t)$ with itself n-times.*

Using Lemma 2.2, Kang-Cho-Jungck [7] have established the following lemma:

LEMMA 2.3. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the conditions* (2.1) *and* (2.2). *Then $\{y_n\}$ defined by* (2.5) *is a Cauchy sequence in X.*

Proof of Theorem 2.1. By Lemma 2.3, $\{y_n\}$ is a Cauchy sequence in X and hence it converges to some point z in X. Consequently, the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converge to z.

Now, suppose that A is continuous. Since A and S are compatible of type (A), it follows from Proposition 1.5 that

$$AAx_{2n} \quad \text{and} \quad SAx_{2n} \ \to \ Az \quad \text{as } n \to \infty.$$

By (2.2), we have

$$d(AAx_{2n}, Bx_{2n-1}) \leq \phi\big(d(AAx_{2n}, SAx_{2n}), d(Bx_{2n-1}, Tx_{2n-1}),$$
$$d(SAx_{2n}, Tx_{2n-1}), d(AAx_{2n}, Tx_{2n-1}),$$
$$d(Bx_{2n-1}, SAx_{2n})\big).$$

Letting $n \to \infty$, we obtain since ϕ is upper-semicontinuous,

$$d(Az, z) \leq \phi\big(0, 0, d(Az, z), d(Az, z), d(Az, z)\big),$$

so that $z = Az$. Since $A(X) \subset T(X)$, there exists a point v in X such that $z = Az = Tv$. Again by using (2.2), we obtain

$$d(AAx_{2n}, Bv) \leq \phi\big(d(AAx_{2n}, SAx_{2n}), d(Bv, Tv),$$
$$d(SAx_{2n}, Tv), d(AAx_{2n}, Tv), d(Bv, SAx_{2n})\big).$$

Letting $n \to \infty$, we have

$$d(z, Bv) \leq \phi\big(0, d(Bv, z), 0, 0, d(Bv, z)\big),$$

which implies that $z = Bv$. Since B and T are compatible mappings of type (A) and $Tv = Bv = z$, by Proposition 1.4, we have $TBv = BTv$ and hence $Tz = Bz$. Moreover, by (2.2), we have

$$d(Ax_{2n}, Bz) \leq \phi\big(d(Ax_{2n}, Sx_{2n}), d(Bz, Tz),$$
$$d(Sx_{2n}, Tz), d(Ax_{2n}, Tz), d(Bz, Sx_{2n})\big).$$

Letting $n \to \infty$, we obtain

$$d(z, Bz) \leq \phi\big(0, 0, d(z, Bz), d(z, Bz), d(Bz, z)\big),$$

so that $z = Bz$ and thus, $Tz = Bz = z$. Since $B(X) \subset S(X)$, there exists a point w in X such that $z = Bz = Sw$. Again using (2.2), we have

$$d(Aw, z) = d(Aw, Bz)$$
$$\leq \phi\big(d(Aw, z), 0, 0, d(Aw, z), 0\big),$$

so that $Aw = z$. Since A and S are compatible mappings of type (A) and $Aw = Sw = z$, we obtain $SAw = ASw$ and hence $Sz = Az$. Therefore, z is a common fixed point of A, B, S and T. Similarly, we can complete the proof when B or S or T is continuous.

Finally, it follows easily from (2.2) that z is a unique common fixed point of A, B, S and T.

The following corollary follows easily from Theorem 2.1:

COROLLARY 2.4. *Let A, B, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions* (2.1), (2.3), (2.4) *and* (2.2) *for all x, y in X, where ϕ satisfies* (i) *and* (iii):

(iii) $\max\{\phi(t,t,t,t,t),\ \phi(t,t,t,2t,0),\ \phi(t,t,t,0,2t)\} < t$ *for each* $t > 0$.

Then A, B, S and T have a unique common fixed point in X.

Now, we show the existence of the common fixed point for compatible mappings of type (A).

EXAMPLE 2.5. Let $X = [0,1]$ with the Euclidean metric d. Define A, B, S and T by

$$Ax = \frac{1}{4}x^{\frac{1}{2}}, \quad Bx = \frac{1}{8}x^{\frac{1}{2}}, \quad Sx = x^{\frac{1}{2}}, \quad \text{and} \quad Tx = \frac{1}{2}x^{\frac{1}{2}}$$

for all x in X. $A(X) \subset [0, \frac{1}{4}] \subset [0, \frac{1}{2}] = T(X)$. Similarly, $B(X) \subset S(X)$. Moreover, by Proposition 1.3 and [7], the pairs A, S and B, T are compatible of type (A). Consider

$$\phi(t_1,t_2,t_3,t_4,t_5) = h\max\{t_1,t_2,t_3,t_4,t_5\}$$

for all t_1, t_2, t_3, t_4 and t_5 in $[0,\infty)$, where $\frac{1}{4} \le h < \frac{1}{2}$. Then ϕ satisfies (i) and (ii) or (iii). Furthermore, we obtain

$$\begin{aligned} d(Ax, By) &= \frac{1}{4}d(Sx, Ty) \\ &\le \phi\big(d(Ax,Sx), d(By,Ty), \\ &\qquad d(Sx,Ty), d(Ax,Ty), d(By,Sx)\big) \end{aligned}$$

for all x, y in X. Thus, all the hypotheses of Theorem 2.1 and Corollary 2.4 are satisfied. Here zero is a unique common fixed point of A, B, S and T.

In the following example, we show that the condition of the compatibility of type (A) is necessary in Theorem 2.1 and Corollary 2.4.

EXAMPLE 2.6. Let $X = [0,\infty)$ with the Euclidean metric d. Define $A = B$ and $S = T : X \to X$ by

$$Ax = \frac{1}{8}x + 1, \quad \text{and} \quad Sx = \frac{1}{2}x + 1$$

for all x in X. Obviously, the sequences $\{Ax_n\}$ and $\{Sx_n\}$ converge to 1 iff $\{x_n\}$ converges to 0 but

$$\lim_{n\to\infty} d(SAx_n, AAx_n) = \frac{3}{8} = \lim_{n\to\infty} d(ASx_n, SSx_n).$$

So, the pair A, S is not compatible of type (A). Consider

$$\phi(t_1, t_2, t_3, t_4, t_5) = h \max\{t_1, t_2, t_3, t_4, t_5\}$$

for all t_1, t_2, t_3, t_4 and t_5 in $[0, \infty)$, where $\frac{1}{4} \leq h < \frac{1}{2}$. Then ϕ satisfies (i) and (ii) or (iii). Furthermore, we have

$$\begin{aligned} d(Ax, Ay) &= \frac{1}{4} d(Sx, Sy) \\ &\leq \phi\big(d(Ax, Sx), d(Ay, Sy), \\ &\qquad d(Sx, Sy), d(Ax, Sy), d(Ay, Sx)\big) \end{aligned}$$

for all x, y in X. We see that all the hypotheses of Theorem 2.1 and Corollary 2.4 are satisfied except the compatibility of type (A) of the pair A, S but A and S do not have a common fixed point in X.

REMARK 2.7. From Theorem 2.1 and Corollary 2.4, we obtain the results of Kang-Cho-Jungck [7], Diviccaro-Sessa [2], Ding [1] by employing compatiblity of type (A) in lieu of compatible, weakly commuting and commuting mappings, respectively. In the sequel, Our theorem also improves a result of Ding [1] by assuming the continuity of any one as opposed to two.

REMARK 2.8. From Theorem 2.1, defining $\phi : [0, \infty)^5 \to [0, \infty)$ by

$$\phi(t_1, t_2, t_3, t_4, t_5) = h \max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\}$$

for all t_1, t_2, t_3, t_4 and t_5 in $[0, \infty)$, where $h \in [0, 1)$ and by employing compatiblity of type (A) in lieu of compatible mappings, we obtain a result of Jungck [4], even if any one of mappings is continuous as opposed to two.

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