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ON THE VALUES OF p—ADIC q—L—FUNCTIONS*

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Throughout this paper $\mathbf{Q}$, $\mathbf{C}$, $\mathbf{Q}_p$ and $\mathbf{C}_p$ will respectively denote the field of rational numbers, the complex number field, the field of p–adic rational numbers and the completion of the algebraic closure of $\mathbf{Q}_p$.

Let v_p be the normalized exponential valuation of $\mathbf{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q–extension, q is variously considered as an indeterminate, a complex number $q \in \mathbf{C}$, or p–adic number $q \in \mathbf{C}_p$. If $q \in \mathbf{C}$, one normally assumes $|q| < 1$. If $q \in \mathbf{C}_p$, one normally assumes $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log_p q)$ for $|x|_p \leq 1$. In the complex case, Carlitz's q–Bernoulli numbers $\beta_k = \beta_k(q)$ can be determined inductively by

$$\beta_0 = 1, q(q\beta+1)^k - \beta^k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

with the usual convention of replacing β^i by β_i.

The q–Bernoulli polynomials are defined by

$$\beta_k(x:q) = (q^x\beta + [x])^k,$$

where $[x] = [x:q] = \frac{1-q^x}{1-q}$. If $q \to 1$, we have the usual Bernoulli numbers and Bernoulli polynomials. Let $F_q(t)$ be generating function of $\beta_k(q)$:

$$F_q(t) = e^{\beta(q)t} = \sum_{k=0}^{\infty} \beta_k(q) \frac{t^k}{k!}.$$

Then this is the unique solution of the following q–difference equation:

$$F_q(t) = qe^t F_q(qt) + 1 - q - t.$$

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In [4], Koblitz constructed a q-analogue of the p-adic L-function $L_{p,q}(s,\chi)$ which interpolates the q-Bernoulli numbers. In section 2, we shall investigate the relation between Carlitz's q-Bernoulli numbers and q-Euler numbers. In section 3 and section 4, we study the values at $s=1$ of the p-adic q-interpolation function $l_{p,q}(u,s:\chi)$ for generalized q-Euler numbers constructed by Satoh [6]. From the relation between $l_{p,q}(u,s:\chi)$ and $L_{p,q}(s,\chi)$, we calculate the values of $L_{p,q}(s,\chi)$ at positive integers.

1. On q-analogue of L-functions

For a primitive Dirichlet character χ with conductor f, the generalized q-Bernoulli numbers $\beta_{k,\chi}(q)$ are defined by

$$\beta_{k,\chi}(q)=[f]^{k-1}\sum_{a=1}^{f}\chi(a)q^{a}\beta_{k}(\frac{a}{f},q^{f}).$$

If $\chi=1$, then we have

$$\beta_{k,\chi}(q)=q,\beta_{k}(1;q)=\begin{cases} q & \text{if } k=0\\ -q\beta_{1}(q) & \text{if } k=1\\ \beta_{k}(q) & \text{if } k>1.\end{cases}$$

This is easily proved [3].

In [1], the q-L-functions are constructed by

$$L_{q}(s,\chi)=\frac{2-s}{s-1}(q-1)\sum_{n=1}^{\infty}\frac{q^{n}\chi(n)}{[n]^{s-1}}+\sum_{n=1}^{\infty}\frac{q^{n}\chi(n)}{[n]^{s}}$$

for $s\in\mathbf{C}$.

Thus we see that

$$L_{q}(1-k,\chi)=-\frac{\beta_{k,\chi}}{k}$$

for any positive integer k.

Let u be complex numbers with $|u| > 1$. Then Carlitz's the q-Euler numbers $H_k(u, q)$ and the q-Euler polynomials $H(u, x : q)$ are defined inductively by

$$H_0(u, q) = 1, (qH + 1)^k - uH_k(u, x : q) = 0 \quad \text{for} \quad k \geq 1$$

with the usual convention of replacing H^k by $H_k(u, q)$ and $H_k(u, x : q) = (q^x H + [x])^k$ for $k \geq 0$.

Since the generalized Carlitz's q-Euler numbers $H_{k,\chi}(u, q)$ are defined by

$$H_{k,\chi}(u, q) = [f]^k \sum_{a=1}^{f} u^{f-a} \chi(a) H_k(u^f, \frac{a}{f} \cdot q^f)$$

for $k \geq 0$.

In [6], the complex function $l_q(u, s : \chi)$ is constructed by

$$l_q(u, s : \chi) = \sum_{n=1}^{\infty} u^{-n} \chi(n) [n]^s$$

for $q \in \mathbf{C}$. This function interpolate the q-Euler numbers as follows [5]:

$$l_q(u, -k : \chi) = \frac{1}{u^f - 1} H_{k,\chi}(u, q).$$

Now we shall investigate the relation between Carlitz's q-Bernoulli numbers and q-Euler numbers. We take d as an element of positive integer with $(d, fp) = 1$ and $\chi(d) \neq 1$. Let ζ_d be primitive d-th root of unity, then we have

$$\begin{aligned}\sum_{j=1}^{d-1} l_p((q\zeta_d^j)^{-1}, s : \chi) &= \sum_{n=1}^{\infty} \frac{q^n \chi(n)}{[n]^s} \sum_{j=1}^{d-1} \zeta_d^{jn} \\ &= -\sum_{n=1}^{\infty} \frac{q^n \chi(n)}{[n]^s} + \sum_{n=1}^{\infty} \frac{q^n \chi(n)}{[n]^s} \sum_{j=0}^{d-1} \zeta_d^{jn} \\ &= -\sum_{n=1}^{\infty} \frac{q^n \chi(n)}{[n]^s} + d[d]^{-s} \chi(d) \sum_{n=1}^{\infty} \frac{q^{dn} \chi(n)}{[n : q^d]^s}.\end{aligned}$$

Thus we have

$$
\begin{aligned}
& -\frac{2-s}{s-1}(q-1)\sum_{j=1}^{d-1} l_q((q\zeta_d^j)^{-1}, s-1 : \chi) - \sum_{j=1}^{d-1} l_q((q\zeta_d^j)^{-1}, s : \chi) \\
&= \frac{2-s}{s-1}(q-1)\sum_{n=1}^{\infty}\frac{q^n\chi(n)}{[n]^{s-1}} - d[d]^{-s+1}\chi(d)\frac{2-s}{s-1}(q-1)\sum_{n=1}^{\infty}\frac{q^{dn}\chi(n)}{[n:q^d]^{s-1}} \\
&+ \sum_{n=1}^{\infty}\frac{q^n\chi(n)}{[n]^s} - d[d]^{-s}\chi(d)\sum_{n=1}^{\infty}\frac{q^{dn}\chi(n)}{[n:q^d]^s} \\
&= L_q(s,\chi) - d[d]^{-s}\chi(d)\Big(\frac{2-s}{s-1}(q^d-1)\sum_{n=1}^{\infty}\frac{q^{dn}\chi(n)}{[n:q^d]^{s-1}} + \sum_{n=1}^{\infty}\frac{q^{dn}\chi(n)}{[n:q^d]^s}\Big) \\
&= L_q(s,\chi) - d[d]^{-s}\chi(d)L_{q^d}(s,\chi).
\end{aligned}
$$

We set $s = 1-k$ for $k > 1$.

$$
\begin{aligned}
& L_q(1-k,\chi) - d[d]^{k-1}\chi(d)L_{q^d}(1-k,\chi) \\
&= \frac{1+k}{k}(q-1)\sum_{j=1}^{d-1} l_q((q\zeta_d^j)^{-1}, -k : \chi) \\
&- \sum_{j=1}^{d-1} l_q((q\zeta_d^j)^{-1}, 1-k : \chi).
\end{aligned}
$$

Thus we obtain the following

LEMMA 1. For $k \in \mathbf{Z}$ with $k > 1$.

$$
\begin{aligned}
& -\frac{\beta_{k,\chi}(q)}{k} + d[d]^{k-1}\chi(d)\frac{\beta_{k,\chi}(q^d)}{k} \\
&= \frac{1+k}{k}(q-1)\sum_{j=1}^{d-1}\frac{1}{((q\zeta_d^j)^{-f}-1)}H_{k,\chi}((q\zeta_d^j)^{-1}, q) \\
&- \sum_{j=1}^{d-1}\frac{1}{((q\zeta_d^j)^{-f}-1)}H_{k-1,\chi}((q\zeta_d^j)^{-1}, q).
\end{aligned}
$$

2. On p-adic q-L-functions

In this section, we assume that q is an element of $\mathbf{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. Let u be an element of $\mathbf{C}_p$ with $|1-u|_p \geq 1$. Then the Euler measure is defined on $\mathbf{Z}_p$ by

$$E_u(a+p^n\mathbf{Z}_p) = \frac{u^{p^n-a}}{1-u^{p^n}}$$

for $a \in \mathbf{Z}$ with $0 \leq a \leq p^n - 1$ and $n \geq 0$. Let f be a positive integer. We denote

$$X = \varprojlim_N (\mathbf{Z}/fp^N\mathbf{Z}).$$

$$X^* = \bigcup_{\substack{0<a<fp \\ (a,p)=1}} a + fp\mathbf{Z}_p,$$

$$a + fp^N\mathbf{Z}_p = \{x \in X | x \equiv a \pmod{fp^N}\},$$

where $a \in \mathbf{Z}$ lies in $0 \leq a < fp^N$.

Note that the natural map

$$\mathbf{Z}/fp^N\mathbf{Z} \to \mathbf{Z}/p^N\mathbf{Z}$$

induces

$$\pi : X \to \mathbf{Z}_p.$$

If g is a function on $\mathbf{Z}_p$, we denote by the same g the function $g \circ \pi$ on X. Namely we consider g as a function on X [2],[3].

We can express the q-Euler numbers as an integral over X, by using the measure E_u, that is.

$$\int_X \chi(x)[x]^k dE_u(x) = \begin{cases} \frac{1}{1-u^f} H_{k,\chi}(u,q) & \text{if } \chi \neq 1 \\ \frac{u}{1-u} H_k(u,q) & \text{if } \chi = 1. \end{cases}$$

Let ω denote the Teichmüller character $\mod p$(if $p = 2$, $\mod 4$). For $x \in X^*$, we set $< x > = < x : q > = \frac{[x]}{\omega(x)}$. Note that $| < x > -1|_p <$

$p^{-\frac{1}{p-1}}$, $< x >^s$ is defined by $\exp(s \log_p < x >)$ for $|s|_p \leq 1$. In [6], an interpolation function $l_{p,q}(u, s : \chi)$ for q–Euler numbers is defined by

$$l_{p,q}(u, s : \chi) = \int_{X^*} < x >^{-s} \chi(x) dE_u(x)$$

for $s \in \mathbf{Z}_p$.

It is known in [6] that

$$\begin{aligned} & l_{p,q}(u, -k : \chi\omega^k) \\ & = \frac{1}{1-u^f} H_{k,\chi}(u, q) - \frac{[p]^k \chi(p)}{1-u^{fp}} H_{k,\chi}(u^p, q^p) \quad \text{if} \quad \chi \neq 1. \end{aligned}$$

We define the operator $\chi^y = \chi^{y,k.q}$ on $f(q)$ by $\chi^y f(q) = [y]^{k-1}\ \chi(y)$ $f(q^y)$, and we define multiplication of these symbols by $\chi^x \chi^y = \chi^{x,k.q^y} \circ \chi^{y,k.q} = \chi^{xy}$.

Koblitz constructed a p–adic q–L–function, for arbitrary fixed $\alpha \in X^*$, as follows:

$$L_{p,q}(1-k, \chi) = -\frac{1}{k}(1 - \chi_k^p)(1 - \frac{1}{\alpha}\chi_k^{\frac{1}{\alpha}})\beta_{k,\chi_k},$$

where $\chi_k = \chi\omega^{-k}(x)$.

Now we refine the above result

$$\begin{aligned} & L_{p,q}(1-k, \chi\omega^k) \\ & = -\frac{1}{k}(\beta_{k,\chi}(q) - [p]^{k-1}\chi(p)\beta_{k,\chi}(q^p) - \frac{1}{\alpha}[\frac{1}{\alpha}]^{k-1}\chi(\frac{1}{\alpha})\beta_{k,\chi}(q^{\frac{1}{\alpha}}) \\ & + \frac{1}{\alpha}[\frac{p}{\alpha}]^{k-1}\chi(\frac{p}{\alpha})\beta_{k,\chi}(q^{\frac{p}{\alpha}})). \end{aligned}$$

Here we shall investigate the relation between $l_{p,q}(u, s : \chi)$ and $L_{p,q}(s, \chi)$. We set $\frac{1}{\alpha} = d$.

$$\begin{aligned} & L_{p,q}(1-k, \chi\omega^k) \\ & = -\frac{1}{k}(\beta_{k,\chi}(q) - [p]^{k-1}\chi(p)\beta_{k,\chi}(q^p) - d[d]^{k-1}\chi(d)\beta_{k,\chi}(q^d)) \\ & + d[pd]^{k-1}\chi(pd)\beta_{k,\chi}(q^{pd}) \\ & = \frac{1}{k}[p]^{k-1}\chi(p)(\beta_{k,\chi}(q^p) - d[d : q^p]^{k-1}\chi(d)\beta_{k,\chi}(q^{pd})) \\ & - \frac{1}{k}(\beta_{k,\chi}(q) - d[d]^{k-1}\chi(d)\beta_{k,\chi}(q^d)). \end{aligned}$$

By using Lemma 1, we have

$$
\begin{aligned}
&L_{p,q}(1-k,\chi\omega^k)\\
&=\frac{1+k}{k}(q-1)\sum_{j=1}^{d-1}\Big(\frac{H_{k,\chi}((q\zeta_d^j)^{-1},q)}{(q\zeta_d^j)^{-f}-1}-\frac{[p]^k\chi(p)H_{k,\chi}((q^p\zeta_d^j)^{-1},q^p)}{(q^p\zeta_d^j)^{-f}-1}\Big)\\
&-\sum_{j=1}^{d-1}\Big(\frac{H_{k-1,\chi}((q\zeta_d^j)^{-1},q)}{(q\zeta_d^j)^{-f}-1}-\frac{[p]^{k-1}\chi(p)H_{k-1,\chi}((q^p\zeta_d^j)^{-1},q^p)}{(q^p\zeta_d^j)^{-f}-1}\Big)\\
&=-\frac{1+k}{k}(q-1)\sum_{j=1}^{d-1}l_{p,q}((q\zeta_d^j)^{-1},-k:\chi\omega^k)\\
&+\sum_{j=1}^{d-1}l_{p,q}((q\zeta_d^j)^{-1},1-k:\chi\omega^k)\\
&=-\frac{1+k}{k}(q-1)\sum_{j=1}^{d-1}\int_{X^*}\chi(x)[x]^k dE_{(q\zeta_d^j)^{-1}}(x)\\
&+\sum_{j=1}^{d-1}\int_{X^*}\chi(x)[x]^{k-1}dE_{(q\zeta_d^j)^{-1}}(x).
\end{aligned}
$$

Since we see that $|1-\zeta_d^{-1}|_p\geq 1$ for $(d,fp)=1$ in [5].

Therefore we obtain the following

PROPOSITION 1.. *For any positive integer k, we have*

$$
\begin{aligned}
&L_{p,q}(1-k,\chi\omega^k)\\
&=-\frac{1+k}{k}(q-1)\sum_{j=1}^{d-1}\int_{X^*}\chi(x)[x]^k dE_{(q\zeta_d^j)^{-1}}(x)\\
&+\sum_{j=1}^{d-1}\int_{X^*}\chi(x)[x]^{k-1}dE_{(q\zeta_d^j)^{-1}}(x).
\end{aligned}
$$

For χ a primitive Dirichlet character mod f, ζ_f a fixed primitive f-th root of unity, $\tau(\chi)=\sum\chi(j)\zeta_f^j$, $\epsilon\neq 1$ a d-th root of unity, and

$(d, fp) = 1$, we have for any continuous $f : X^* \to \mathbf{C}_p$ [5]

$$\int_{X^*} \chi(x) f(x) d\mu_\epsilon(x) = \frac{\tau(x)}{f} \sum_{0 \le a < f} \bar{\chi}(a) \int_{X^*} f(x) d\mu_{\zeta_f^{-a}\epsilon}(x).$$

Let $f(x) = [x]^k$ and $\epsilon = \zeta_d^{-1}$. Thus we have

$$\begin{aligned} &L_{p,q}(1-k, \chi\omega^k) \\ &= -\frac{1+k}{k}(q-1) \sum_{j=1}^{d-1} \int_{X^*} \chi(x)[x]^k dE_{(q\zeta_d^j)^{-1}}(x) \\ &+ \sum_{j=1}^{d-1} \int_{X^*} \chi(x)[x]^{k-1} dE_{(q\zeta_d^j)^{-1}}(x) \\ &= -(\frac{1+k}{k}(q-1)\frac{\tau(x)}{f} \sum_{0 \le a < f} \bar{\chi}(a) \sum_{j=1}^{d-1} \int_{X^*} [x]^k dE_{\zeta_f^{-a}(q\zeta_d^j)^{-1}}(x)) \\ &+ \frac{\tau(x)}{f} \sum_{0 \le a < f} \bar{\chi}(a) \sum_{j=1}^{d-1} \int_{X^*} [x]^{k-1} dE_{\zeta_f^{-a}(q\zeta_d^j)^{-1}}(x)). \end{aligned}$$

PROPOSITION 2.. *For any positive integer k,*

$$\begin{aligned} &L_{p,q}(1-k, \chi\omega^k) \\ &= -(\frac{1+k}{k}(q-1)\frac{\tau(x)}{f} \sum_{0 \le a < f} \bar{\chi}(a) \sum_{j=1}^{d-1} \int_{X^*} [x]^k dE_{\zeta_f^{-a}(q\zeta_d^j)^{-1}}(x)) \\ &+ \frac{\tau(x)}{f} \sum_{0 \le a < f} \bar{\chi}(a) \sum_{j=1}^{d-1} \int_{X^*} [x]^{k-1} dE_{\zeta_f^{-a}(q\zeta_d^j)^{-1}}(x)). \end{aligned}$$

3. The value at $s = 1$ of $l_{p,q}(\zeta_d^{-1}, s : \chi)$

$l_{p,q}(u, s : \chi)$ is constructed in [6]. In particular we take $u = \zeta_d^{-1}$. Then

$$
\begin{aligned}
&l_{p,q}(\zeta_d^{-1}, 1 - k : \chi\omega^k) \\
&= \int_{X^*} [x]^{k-1}\chi(x) dE_{\zeta_d^{-1}}(x) \\
&\frac{\tau(x)}{f} \sum_{0 \le a < f} \bar{\chi}(a) \int_{X^*} [x]^{k-1} dE_{\zeta_d^{-1}\zeta_f^{-a}}(x).
\end{aligned}
$$

Thus we have

$$
l_{p,q}(\zeta_d^{-1}, 1 : \chi) = \frac{\tau(x)}{f} \sum_{0 \le a < f} \bar{\chi}(a) \int_{X^*} [x]^{k-1} dE_{\zeta_d^{-1}\zeta_f^{-a}}(x).
$$

Now we define the function $\log_{[p]}$ as follows:

$$
\log_{[p]}(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{[n]}.
$$

It is easily proved in [5] that

$$
\begin{aligned}
&\int_{X^*} \frac{1}{[x]} dE_{\zeta_d^{-1}\zeta_f^{-a}}(x) \\
&= -\log_{[p]}(1 - \zeta_d^{-1}\zeta_f^{-a}) + \frac{1}{[p]} \log_{[p, q^p]}(1 - (\zeta_d^{-1}\zeta_f^{-a})^p).
\end{aligned}
$$

Therefore we obtain the following

THEOREM 1. *For χ a primitive Dirichlet (mod f), ζ_f a fixed primitive f–th root of unity, $\tau(\chi) = \sum \chi(j)\zeta_f^j$, $\zeta_d \neq 1$ a d–th root of unity, and $(d, fp) = 1$, we have*

$$
\begin{aligned}
&l_{p,q}(\zeta_d^{-1}, 1 : \chi) \\
&= \frac{\tau(x)}{f} \sum_{0 \le a < f} \bar{\chi}(a)(-\log_{[p]}(1 - \zeta_d^{-1}\zeta_f^{-a}) + \frac{1}{[p]} \log_{[p, q^p]}(1 - (\zeta_d^{-1}\zeta_f^{-a})^p)).
\end{aligned}
$$

In particular

$$
\lim_{q \to 1} l_{p,q}(\zeta_d^{-1}, 1, \chi) = l_p(\zeta_d^{-1}, 1, \chi).
$$

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