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IRREDUCIBLE MODULES FOR SOME METACYCLIC GROUPS

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The aim of this note is to give an explicit description of all isomorphism types of irreducible modules over a finite field for a metacyclic group presented by $\langle x, y \mid x^m = 1,\ y^q = 1,\ y^{-1}xy = x^r \rangle$ where q is a prime and r is a q-th roots of 1 modulo m. The main results of this note generalize the invesigation by Barlotti [1] for metacyclic groups of order pq (p, q . primes).

1. Background Results

We first set up some notation which will be kept throughout this note. Let $\mathbb{F}$ be a finite field and let $a(n)$ denote the multiplicative order of $|\mathbb{F}|$ modulo n for every positive integer n. Let $G(m,n)$ be a metacyclic group defined by

$$G(m,n) = \langle x, y \mid x^m = 1,\ y^n = 1,\ y^{-1}xy = x^r \rangle$$

where r is a primitive n-th root of 1 modulo m; note that all possible such r give the same group for the fixed integers m and n. When n is a prime, for each positive divisor d of m the group defined by

$$\langle x, y \mid x^d = 1,\ y^n = 1,\ y^{-1}xy = x^r \rangle$$

is $G(d,n)$ provided that d does not divide $r-1$, while the group is abelian if d divides $r-1$.

Most of notation and terminology which are not defined in this note are standard, or can be found in [3] or [2].

We continue with some important construction of faithful irreducible modules for the group $G(m,n)$.

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CONSTRUCTION 1.1. *Let m be a positive integer not divisible by the characteristic of $\mathbb{F}$ and let n be a divisor of $a(m)$. Let $\mathbb{K}$ be the field with $|\mathbb{F}|^{a(m)}$ elements, let u be an element of multiplicative order m in $\mathbb{K}$, and write V for the $\mathbb{K}$ viewed as a vector space over $\mathbb{F}$.*

(a) There is an action of $G(m,n)$ on V such that for every v in V,

$$vx = vu \quad \text{and} \quad vy = v^{|\mathbb{F}|^{a(m)/n}}.$$

(b) Under the action in (a), V is a faithful irreducible module for $G(m,n)$ over $\mathbb{F}$; denote the module by $V(u)$.

(c) $End_{\mathbb{F}G(m,n)}V(u)$ is the field with $|\mathbb{F}|^{a(m)/n}$ elements; in particular, if $n = a(m)$ then $V(u)$ is an absolutely irreducible faithful module for $G(m,n)$ over $\mathbb{F}$.

Note that this construction is well known for $d = 1$ from the representation theory of cyclic groups, while the proof for the general case can be found in [1]. It is also well known that every faithful irreducible module for a finite cyclic group (say, G(m,1) here) is realized as such a module described in this construction, and $V(u)$ and $V(v)$ are isomorphic if and only if u and v are roots of the same irreducible factor of $x^m - 1$ in $\mathbb{F}[x]$.

Let A be a finite abelian group and V an irreducible $\mathbb{F}A$-module. The factor group of A by the kernel $\{g \in A : vg = v \text{ for all } v \in V\}$ of V is cyclic. Conversely, every subgroup of A with cyclic quotient becomes the kernel of a certain irreducible $\mathbb{F}A$-module, provided that the characteristic of $\mathbb{F}$ does not divide the order of the cyclic quotient. This leads to a complete description of the irreducible modules for a finite abelian group over a finite field.

Suppose that the abelian group A is metacyclic. Then A is a direct product of two finite cyclic groups C_m and C_n for some nonnegative integers m and n such that n divides m. For any positive divisor d of m we define $\#(d)$ to be the number of all cyclic quotients of order d of A. If the characteristic of $\mathbb{F}$ does not divide m, there exists precisely $\sum_{d|m} \#(d) \cdot \phi(d)/a(d)$ pairwise nonisomorphic irreducible modules for A over $\mathbb{F}$.

2. Main Results

Let p be the characteristic of $\mathbb{F}$, let q be a fixed prime, let m be a fixed positive integer and let d be a positive divisor of m. Let G be a finite group whose factor group by the largest normal p-subgroup $O_p(G)$ is isomorphic to $G(m,q)$. Since $O_p(G)$ is contained in the kernels of all irreducible $\mathbb{F}G$-modules, there is a natural one-to-one correspondence between the irreducible $\mathbb{F}G$-modules and the irreducible $\mathbb{F}G(m,q)$-modules.

We now consider faithful irreducible modules for $G(m,q)$ whose order is not divisible by p. The cyclic normal subgroup generated by x in $G(m,q)$ is denoted by M.

THEOREM 2.1. *If q divides $a(m)$, every faithful irreducible module for $G(m,q)$ over $\mathbb{F}$ is isomorphic to an $\mathbb{F}G(m,q)$-module described in Construction 1.1 So there exist precisely $\phi(m)/a(m)$ isomorphism types of faithful irreducible modules for $G(m,q)$ over $\mathbb{F}$.*

Proof. Let $V_1,...,V_n$ be pairwise nonisomorphic faithful irreducible modules for M over $\mathbb{F}$, where $n = \phi(m)/a(m)$. Let $W_1,...,W_n$ be the faithful irreducible modules for $G(m,q)$ over $\mathbb{F}$, as described in Construction 1.1, such that $(W_i)_M \cong V_i$ for all $i = 1,...,n$ Then $\mathbb{F}M = V_0 \oplus V_1 \oplus \cdots \oplus V_n$ for some $\mathbb{F}M$-module V_0. It follows that $\mathbb{F}G(m,q) \cong V_0^{G(m,q)} \oplus V_1^{G(m,q)} \oplus \cdots \oplus V_n^{G(m,q)}$. For each $i = 1,...,n$, the multiplicity of W_i as a composition factor in the head of $V_i^{G(m,q)}$ is $(\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}M} V_i)/(\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}G(m,q)} W_i) = a(m)/(a(m)/q) = q$ by Construction 1.1 (c) and Theorem 4.13 in [2]. Therefore, $V_i^{G(m,q)}$ is isomorphic to the direct sum of q copies of W_i.

Let W be a irreducible $\mathbb{F}G(m,q)$-module which is not isomorphic to W_i for all $i = 1,...,n$. Then W is a homomorphic image of $V^{G(m,q)}$ for some irreducible submodule V of V_0, and hence V is isomorphic to a submodule of W_M. It follows that $\mathrm{Ker}\, W \geqslant \mathrm{Ker}\, V^{G(m,q)} = \mathrm{Core}_{G(m,q)} \mathrm{Ker}\, V = \mathrm{Ker}\, V \neq 1$, which implies W is not faithful. Consequently, every faithful irreducible module for $G(m,q)$ over $\mathbb{F}$ is isomorphic to one of the W_i. □

LEMMA 2.2. *Let V a faithful irreducible module for M over $\mathbb{F}$. If q does not divide $a(m)$, then V is not isomorphic to $V \otimes y$.*

Proof. There are precisely $\phi(m)/a(m)$ isomorphism types of faithful irreducible modules for M over $\mathbb{F}$, which are transitively permuted by

Aut M. It follows that the stabilizer in Aut M of the isomorphism type of V is a subgroup of index $\phi(m)/a(m)$ in Aut M (equivalently, of order $a(m)$).

The statement $V \cong_{\mathbb{F}M} V \otimes y$ says that the element which maps x to x^r (of order q) in Aut M lies in this subgroup of order $a(m)$. It follows that $V \cong_{\mathbb{F}M} V \otimes y$ implies $y \mid a(m)$. □

THEOREM 2.3. *If q does not divide $a(m)$, then*

(a) every $\mathbb{F}G(m,q)$-module induced from a faithful irreducible module for M over $\mathbb{F}$ is faithful and irreducible;

(b) every faithful irreducible module for $G(m,q)$ over $\mathbb{F}$ is induced from a faithful irreducible module for M over $\mathbb{F}$.

Proof. (a) Let V be a faithful irreducible module for M over $\mathbb{F}$. Then $V^{G(m,q)}$ is faithful, since the kernel of $V^{G(m,q)}$ is the core of the kernel of V in $G(m,q)$. By Lemma 2.2 and Theorem 9.6 b) in [3], $V^{G(m,q)}$ is irreducible.

(b) Let $V_1, ..., V_n$ be the $\phi(m)/a(m)$ pairwise nonisomorphic faithful irreducible modules for M over $\mathbb{F}$. Suppose $\mathbb{F}M = V_0 \oplus V_1 \oplus \cdots \oplus V_n$. Then $\mathbb{F}G(m,q) \cong V_0^{G(m,q)} \oplus V_1^{G(m,q)} \oplus \cdots \oplus V_n^{G(m,q)}$. No irreducible constituent of V_0 is faithful, so every faithful irreducible module for $G(m,q)$ over $\mathbb{F}$ is isomorphic to one of the $V_i^{G(m,q)}$. □

COROLLARY 2.4. *Assume that the characteristic of $\mathbb{F}$ does not divide d. There exist precisely $\phi(d)/[a(d),q]$ isomorphism types of faithful irreducible modules for $G(d,q)$ over $\mathbb{F}$, where $[a(d),q]$ is the least common multiple of $a(d)$ and q*

Proof. If q divides $a(d)$, then from Theorem 2.1, there exist precisely $\phi(d)/a(d)$ isomorphism types of faithful irreducible modules for $G(d,q)$ over $\mathbb{F}$.

If q does not divide $a(d)$, then $V_i \ncong V_i \times y^j$ for all $j = 0, ..., q-1$, by Lemma 2.2. Since $V_i^{G(m,q)} \cong V_j^{G(m,q)}$ if and only if $V_i \cong V_j \otimes y^k$ for some $k = 0, ..., q-1$, the multiplicity of V_i as a composition factor in $V_1^{G(m,q)} \oplus \cdots \oplus V_n^{G(m,q)}$ is q for all $i = 1, ..., n$. Hence there are exactly $\phi(d)/a(d)q$ isomorphism types of faithful irreducible modules for $G(d,q)$over $\mathbb{F}$ □

Let d_0 be the greatest common divisor m and $r-1$, and let Δ be the set of all positive divisors of m which do not divide $r-1$. Then we have

THEOREM 2.5. *Let G be a finite group whose factor group by the largest normal p-subgroup is isomorphic to $G(m,q)$. There is a one-to-one correspondence between the set of isomorphism types of all irreducible $\mathbb{F}G$-modules and the union of the following two sets: (i) the set of isomorphism types of all faithful irreducible $\mathbb{F}G(d,q)$-modules, where d runs through Δ, (ii) the set of isomorphism types of all irreducible $\mathbb{F}(C_{d_0} \times C_q)$-modules.*

Proof. Suppose N is a normal subgroup of $G(m,q)$. If N contains the commutator subgroup $G(m,q)'$ then $G(m,q)/N$ is abelian; otherwise, N is contained in M, so $G(m,q)/N \cong G(d,q)$ for some d in Δ. On the other hand, for each d in Δ there exists a unique normal subgroup N such that $G(d,q) \cong G(m,q)/N$. Since $G(m,q)' = \langle x^{r-1} \rangle$ it follows easily that $G(m,q)/G(m,q)' \cong C_{d_0} \times C_q$, and hence the theorem is proven. □

References

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