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ON CERTAIN AREA FUNCTIONS ASSOCIATED WITH APPROACH REGIONS

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1. Introduction

In this paper we first define a group of homogeneous type G, which is a more general setting than $\mathbf{R}^n$, and we also consider the space $G \times (0, \infty)$, which is a kind of generalized upper half-space over G. Then we shall assume that to each boundary point $x \in G$, there is associated an approach region $\Gamma_\alpha(x) \subset G \times (0, \infty)$. Let f be a function defined on $G \times (0, \infty)$. For $x \in G$ and $\alpha > 0$, we define an area function $S_\alpha(f)$ associated with $\Gamma_\alpha(x)$, by

$$S_\alpha(f)(x) = \left(\int_{\Gamma_\alpha(x)} |f(y,t)|^2 \frac{d\mu(y)dt}{t^{n+1}} \right)^{1/2},$$

where μ denotes the Borel measure on G. For simplicity, we put $S(f) = S_1(f)$. The purpose of this paper is to study inequalities for the L^p norms of area functions $S_\alpha(f)$ and $S(f)$ for $\alpha > 1$; more precisely, let $0 < p < \infty$ and $\alpha > 1$, then there is a constant C such that

$$\|S_\alpha(f)\|_{L^p(d\mu)} \leq C \|S(f)\|_{L^p(d\mu)}.$$

Throughout this paper we shall use the letter C to denote a constant which need not be the same at each occurrence.

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2.Preliminaries

Let G be a topological group. Assume that d is a pseudo-distance on G, i.e., a nonnegative function defined on $G \times G$ with properties

(i) $d(x,x) = 0; d(x,y) > 0$ if $x \neq y$,
(ii) $d(x,y) = d(y,x)$, and
(iii) there is a constant K such that

$$d(x,z) \leq K[d(x,y) + d(y,z)] \quad \text{for all} x,y,z \in G.$$

Assume also that

(a) the balls $B(x,\rho) = \{y \in G : d(x,y) < \rho\}$, $\rho > 0$, form a basis of open neighborhoods at $x \in G$,

and that μ is a Borel measure on G, and

(b) there is a constant A such that

$$0 < \mu(B(x,2\rho)) \leq A\mu(B(x,\rho)) < \infty \quad \text{for all} x \in G, \rho > 0.$$

Assume further that μ is left-invariant:

(c) $\mu(xE) = \mu(E)$ for $x \in G$, measurable $E \subset G$, and
(d) $\mu(E^{-1}) = \mu(E)$,

and that d is left-invariant:

(e) $xB(y,\rho) = B(xy,\rho)$ for all $x, y \in G$, $\rho > 0$.

Then we call (G,d,μ) a *group of homogeneous type*. Let (G,d,μ) be a group of homogeneous type and $\rho > 0$. Then an automorphism δ_ρ of G is called a *dilation* of G if there is a positive integer n such that

$$\mu(\delta_\rho(E)) = \rho^n \mu(E) \tag{1}$$

for any measurable $E \subset G$, and in particular,

$$\mu(\delta_\rho(B(e,1))) = \mu(B(e,\rho)) = C_n \rho^n, \tag{2}$$

where C_n denotes the volume of the unit ball $B(e,1)$, and e denotes the identity element of G. Frequently, we shall write ρx instead of $\delta_\rho x$

for $\rho > 0$ and $x \in G$. For details see [6]. A. Korányi and S. Vági [3] studied that d is left-invariant if and only if

$$d(x, y) = |x^{-1}y|,$$

where $|\cdot|$ is a nonnegative function on G with properties

(i) $|x| = 0$ if and only if $x = e$,
(ii) there is a constant K such that $|xy| \leq K(|x| + |y|)$, and
(iii) $|x^{-1}| = |x|$.

For $x, y \in G$ and $\rho > 0$, the set

$$B(x, \rho) = \{y \in G : |x^{-1}y| < \rho\}$$

is called the *ball* centered at $x \in G$ with radius ρ. Now consider the space $G \times (0, \infty)$, which is a kind of generalized upper half-space over G We then introduce the analogue of nontangential or conical region. For $x \in G$ and $\alpha > 0$, set

$$\Gamma_\alpha(x) = \{(y, t) \in G \times (0, \infty) : |x^{-1}y| < \alpha t\}.$$

For simplicity, we put $\Gamma(x) = \Gamma_1(x)$. For any closed subset $F \subset G$ and $\alpha > 0$, set

$$\mathcal{R}_\alpha(F) = \bigcup_{x \in F} \Gamma_\alpha(x).$$

Then the *tent* over an open subset $O = {}^cF$ of G, denoted by $T(O)$, is given as

$$T(O) = {}^c\mathcal{R}_1(F).$$

We define an area function associated with an approach region as follows. Let f be a function defined on $G \times (0, \infty)$. For $x \in G$ and $\alpha > 0$, set

$$S_\alpha(f)(x) = \left(\int_{\Gamma_\alpha(x)} |f(y, t)|^2 \frac{d\mu(y)dt}{t^{n+1}} \right)^{1/2}.$$

Let f be a locally integrable function on G. For $x \in G$, we define

$$M(f)(x) = \sup_{x \in B} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y),$$

where the supremum is taken over all balls B containing x. Then $M(f)$ is called the *Hardy-Littlewood maximal function* of f. We need the notion of points of density. Suppose F is a closed subset of G and γ is a fixed parameter, $0 < \gamma < 1$. Then we say t hat a point $x \in G$ has *global γ-density* with respect to F, if

$$\frac{\mu(F \cap B)}{\mu(B)} \geq \gamma$$

for *all* balls B centered at x in G. Let F^* be the set of points of global γ-density with respect to F; then F^* is closed, $F^* \subset F$, and

$${}^cF^* = \{x \in G : M(\chi_{{}^cF})(x) > 1 - \gamma\},$$

where $\chi_{{}^cF}$ is the characteristic function of the open set ${}^cF$.

3.Main result

We state the four lemmas we need.

LEMMA 1 [6]. *Assume F is a closed subset of G. Then there is a constant C such that*

$$\mu({}^cF^*) \leq C\mu({}^cF),$$

where F^* is the set of points of global γ-density with respect to F.

LEMMA 2. *Suppose $\alpha > 0$ is given Then there is a constant C so that whenever F is a closed subset of G and $A(y,t)$ is any nonnegative measurable function on $G \times (0,\infty)$, then*

$$\int_F \left(\int_{\Gamma_\alpha(x)} A(y,t) d\mu(y) dt \right) d\mu(x) \leq C \int_{\mathcal{R}_\alpha(F)} A(y,t) t^n d\mu(y) dt.$$

Proof. Fubini's theorem gives

$$\int_F \left(\int_{\Gamma_\alpha(x)} A(y,t) d\mu(y) dt \right) d\mu(x)$$
$$= \int_{G \times (0,\infty)} A(y,t) \left(\int_F \chi_{B(y,\alpha t)}(x) d\mu(x) \right) d\mu(y) dt,$$

and so, for given $(y,t) \in \mathcal{R}_\alpha(F)$, it will suffice to show that there is a constant C so that

$$\int_F \chi_{B(y,\alpha t)}(x)d\mu(x) \le Ct^n.$$

In fact, let $(y,t) \in \mathcal{R}_\alpha(F)$. Then

$$\begin{aligned}\int_F \chi_{B(y,\alpha t)}(x)d\mu(x) &\le \int_G \chi_{B(y,\alpha t)}(x)d\mu(x)\\ &= Ct^n,\end{aligned}$$

as desired. The proof is therefore complete.

LEMMA 3 [6]. *Suppose $\alpha > 0$ is given Then there are constants C and γ, $0 < \gamma < 1$, sufficiently close to 1, so that whenever F is a closed subset of G and $A(y,t)$ is a nonnegative measurable function on $G \times (0,\infty)$, then*

$$\int_{\mathcal{R}_\alpha(F^*)} A(y,t)t^n d\mu(y)dt \le C\int_F \left(\int_{\Gamma(x)} A(y,t)d\mu(y)dt\right) d\mu(x),$$

where F^* is the set of points of global γ-density with respect to F.

LEMMA 4. *Let f be a nonnegative function defined on G. Suppose*

$$\Lambda_t(f)(x) = \frac{1}{t^n}\int_G \chi_{B(y,t)}(x)f(x)d\mu(x), \quad t > 0.$$

Then there is a constant C such that

$$\Lambda_{\alpha t}(f) \le C\Lambda_t(M(f)).$$

Proof. If $f \ge 0$, then

$$\begin{aligned}\Lambda_{\alpha t}(f) &\le C\Lambda_t(\Lambda_{\alpha t}(f))\\ &\le C\Lambda_t(M(f)),\end{aligned}$$

since $\Lambda_{\alpha t}(f) \le CM(f)$. The proof is therefore complete.

Our main result is the following:

THEOREM 5. *Let $0 < p < \infty$ and $\alpha > 1$. Then there is a constant C such that*

$$||S_\alpha(f)||_{L^p(d\mu)} \le C||S(f)||_{L^p(d\mu)},$$

with $S(f) = S_1(f)$.

Proof. Assume first that $0 < p < 2$. For each $\lambda > 0$, we define the open set O by

$$O = {}^cF = \{x \in G : S(f)(x) > \lambda\}.$$

Let $O^* = {}^cF^*$. Then we take $F^* \subset F$ to be the set of points of global γ-density with respect to F. Apply Lemma 2 with $A(y,t) = |f(y,t)|^2 t^{-n-1}$ (and F^* in place of F), and we obtain

$$\int_{F^*} S_\alpha(f)(x)^2 d\mu(x) \le C \int_{\mathcal{R}_\alpha(F^*)} |f(y,t)|^2 \frac{d\mu(y)dt}{t}. \tag{3}$$

Next apply Lemma 3, again with $A(y,t) = |f(y,t)|^2 t^{-n-1}$, and we obtain

$$\begin{aligned} &\int_{\mathcal{R}_\alpha(F^*)} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \\ &\le C \int_F \left(\int_{\Gamma(x)} |f(y,t)|^2 \frac{d\mu(y)dt}{t^{n+1}} \right) d\mu(x). \end{aligned} \tag{4}$$

Then (3) and (4) imply that

$$\int_{F^*} S_\alpha(f)(x)^2 d\mu(x) \le C \int_F S(f)(x)^2 d\mu(x). \tag{5}$$

Thus it follows from Lemma 1 and (5) that

$$\begin{aligned} &\mu(\{x \in G \cdot S_\alpha(f)(x) > \lambda\}) \\ &\le \mu(O^*) + \frac{C}{\lambda^2} \int_F S(f)(x)^2 d\mu(x) \\ &\le C \left(\mu(O) + \frac{1}{\lambda^2} \int_F S(f)(x)^2 d\mu(x) \right) \\ &= C \left(\mu(\{x \in G : S(f)(x) > \lambda\}) + \frac{1}{\lambda^2} \int_F S(f)(x)^2 d\mu(x) \right). \end{aligned} \tag{6}$$

Multiply both sides of (6) by λ^{p-1} and integrate, then we get that

$$\|S_\alpha(f)\|_{L^p(d\mu)} \le C\|S(f)\|_{L^p(d\mu)}$$

for $0 < p < 2$. Assume second that $2 \le p < \infty$. Then observe that

$$\|S_\alpha(f)\|^2_{L^p(d\mu)} = \sup_\psi \int_G S_\alpha(f)(x)^2\psi(x)d\mu(x),$$

where the supremum is taken over all ψ which belong to $L^r(d\mu)$ with r dual to $2/p$, and $\|\psi\|_{L^r(d\mu)} \le 1$. Then it follows from Lemma 4 that

$$\begin{aligned}
(7) \qquad & \int_G S_\alpha(f)(x)^2\psi(x)d\mu(x) \\
&= \int_{G\times(0,\infty)} |f(y,t)|^2\Lambda_{\alpha t}(\psi)(y)\frac{d\mu(y)dt}{t} \\
&\le C\int_{G\times(0,\infty)} |f(y,t)|^2\Lambda_t(M(\psi))(y)\frac{d\mu(y)dt}{t} \\
&= C\int_G S(f)(x)^2 M(\psi)(x)d\mu(x) \\
&\le C\|S(f)\|^2_{L^p(d\mu)}\|M(\psi)\|_{L^r(d\mu)} \\
&\le C\|S(f)\|^2_{L^p(d\mu)}\|\psi\|_{L^r(d\mu)} \qquad (\text{by } r > 1) \\
&\le C\|S(f)\|^2_{L^p(d\mu)}.
\end{aligned}$$

Taking the supremum over all ψ in (7). Then

$$\|S_\alpha(f)\|_{L^p(d\mu)} \le C\|S(f)\|_{L^p(d\mu)}$$

with $2 \le p < \infty$. The proof is therefore complete.

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