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ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS IN GENERAL BANACH SPACES

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1. Introduction and Preliminaries

Let X be a Banach space with norm $\|\cdot\|$ and ω be a real number. We let C denote the space of all continuous function $\psi : [-r, 0] \to X$ for fixed $r > 0$. For $\psi \in C$, $\|\psi\|_C = \sup_{-r \le \theta \le 0} \|\psi(\theta)\|$. We denote the symbol x_t by $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$.

We consider the abstract functional differential equation of the type

$$\text{(FDE)} \qquad \begin{aligned} &x'(t) \in A(t, x_t)x(t) + G(t, x_t), \quad 0 \le t \le T, \\ &x_0 = \phi, \quad -r \le t \le 0, \end{aligned}$$

in a general Banach space. An operator $A : D \subset X \to 2^X$ is called dissipative if

$$< y_1 - y_2, x_1 - x_2 >_i \le 0$$

for every $[x_1, y_1], [x_2, y_2] \in A$. A is said to be ω-dissipative if $A - \omega I$ is dissipative. A is called m-dissipative if A is dissipative and $R(I - \lambda A) = X$ for all $\lambda > 0$. We recall that

$$< y, x >_s = \|x\| < y, x >_+ = \lim_{t \to 0+} \frac{\|x + ty\| - \|x\|}{t}$$

and

$$< y, x >_i = \|x\| < y, x >_- = \lim_{t \to 0-} \frac{\|x + ty\| - \|x\|}{t}.$$

For other properties of $< \cdot, \cdot >_i$ and $< \cdot, \cdot >_s$, we refer the reader to Lakshmikantham and Leela([9]) and Kobayashi([8]). Pavel([10]) has

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obtained the existence of a unique limit solution for the abstract nonlinear evolution equation of the type

$$x'(t) \in A(t)x(t), x(t_0) = x_0, x_0 \in X, 0 \le t_0 \le t \le T$$

in a general Banach space by the constructing the step functions which satisfies certain approximate discrete scheme. Kartsatos([4]) and Kartsatos and Parrott([5]) have studied nonlinear functional evolution equations by making use of fixed theoerm and Crandall and pazy's result([2]). In this paper, we have obtained the existence of a local limit solution of (FDE) by the virtue of the limit solution for (FDE) with fixed functional term in A and Banach contraction principle.

We need the following hypotheses in our subsequent discussion
(H.1) For each $(t, \psi) \in [0, T] \times C$ and $0 < \lambda < \lambda_0, \lambda_0\omega < 1$, $A(t, \psi) : D(A(t, \psi)) \subset X \to 2^X$ is ω-dissipative in X and $R(I - \lambda A(t, \psi)) = X$, where $D(A(t, \psi))$ is only dependent on t. We denote $D(A(t, \psi)) = D(t)$.
(H.2) There exists a nondecreasing continuous function $L : \mathcal{R}^+ \times \mathcal{R}^+ \times \mathcal{R}^+ \to \mathcal{R}^+ = [0, \infty)$ such that

$$\begin{aligned} &< y_1 - y_2, x_1 - x_2 >_i \\ &\qquad \le \omega\|x_1 - x_2\|^2 + L(\|\psi_1\|_C, \|\psi_2\|_C, \min\{\|x_1\|, \|x_2\|\}) \\ &\qquad \cdot [|t - s|(1 + |A(s, \psi_2)x_2|) + \|\psi_1 - \psi_2\|_C]\|x_1 - x_2\|, \end{aligned}$$

for all $0 \le s \le t \le T$, $[x_1, y_1] \in A(t, \psi_1)$ and $[x_2, y_2] \in A(s, \psi_2)$, where $|A(s, \psi_2)x_2| = \inf\{\|y\| \,|\, y \in A(s, \psi_2)x_2\}$.
(H.3) There exist $\beta > 0$ and a nondecreasing continuous function $L_1 : \mathcal{R}^+ \to \mathcal{R}^+$ such that

$$\|G(t, \psi_1) - G(t, \psi_2)\| \le \beta\|\psi_1 - \psi_2\|_C,$$
$$\|G(t, \psi) - G(s, \psi)\| \le L_1(\|\psi\|_C)|t - s|,$$

for each $t, s \in [0, T]$, and $\psi, \psi_1, \psi_2 \in C$.
(H.4) ϕ is a given Lipshitz function with Lipschitz constant L_0 on $[-r, 0]$ and $\phi(0) \in D(A(0, \phi))$ for every $t \in [0, T]$.

Let

$$M = \max\{L_0, 1 + \beta 4\|\phi\|_C + \|G(0, \bar{0})\| + |A(0, \phi)\phi(0)|\}.$$

We define a set E by

$$E = \{u : [-r, T] \to X \mid u(t) \text{ is continuous, } u(t) = \phi(t) \text{ for } t \in [-r, 0] \text{ and } \|u(t_1) - u(t_2)\| \le M|t_1 - t_2| \text{ for } t_1, t_2 \in [0, T]\}.$$

Clearly $E \neq \phi$ since the function defined by $u(t) = \phi(t)$ for $t \in [-r, 0]$, and $u(t) = \phi(0)$ for $t \in [0, T]$ belongs to E. Moreover, the set E is a complete metric space with supremum norm.

Modifying Lemma 2.1 in [11],the basic properties are given by

LEMMA 1.1.

(1) *(H.2) is equivalent with*

(1.1)

$$\begin{aligned}&(1 - \lambda\omega)\|x_1 - x_2\| \\ &\le \|x_1 - x_2 - \lambda(y_1 - y_2)\| + \lambda L(\|\psi_1\|_C, \|\psi_2\|_C, \min\{\|x_1\|, \|x_2\|\}) \\ &\quad \cdot [|t - s|(1 + |A(s, \psi_2)x_2|) + \|\psi_1 - \psi_2\|_C]\end{aligned}$$

for all $\lambda > 0$, $0 \le s \le t \le T$, $[x_1, y_1] \in A(t, \psi_1)$, $[x_2, y_2] \in A(s, \psi_2)$.

(2) *(1.1) implies*

$$\begin{aligned}&(\lambda + \mu - \lambda\mu\omega)\|x_1 - x_2\| \le \lambda\|x_2 - \mu y_2 - x_1\| \\ &+ \mu\|x_1 - \lambda y_1 - x_2\| + \lambda\mu L(\|\psi_1\|_C, \|\psi_2\|_C, \min\{\|x_1\|, \|x_2\|\}) \\ &\quad \cdot [|t - s|(1 + |A(s, \psi_2)x_2|) + \|\psi_1 - \psi_2\|_C]\end{aligned} \tag{1.2}$$

for all $\lambda, \mu > 0$, $0 \le s \le t \le T$, $[x_1, y_1] \in A(t, \psi_1)$, $[x_2, y_2] \in A(s, \psi_2)$ *and the above ineqaulity implies*

$$\begin{aligned}&(1 - \lambda\omega)\|x_1 - \bar{u}\| \\ &\le \|x_1 - \lambda y_1 - \bar{u}\| + \lambda|A(s, \psi_2)\bar{u}| + \lambda L(\|\psi_1\|_C, \|\psi_2\|_C, \|\bar{u}\|) \\ &\quad \cdot [|t - s|(1 + |A(s, \psi_2)x_2|) + \|\psi_1 - \psi_2\|_C]\end{aligned} \tag{1.3}$$

for all $\lambda > 0$, $0 \le s \le t \le T$, $[x_1, y_1] \in A(t, \psi_1)$, $\bar{u} \in D(A(s, \psi_2))$

In view of (H.1) and (H.2), $\overline{D(A(t, \psi))}$ *is independent of* (t, ψ) *for each* $(t, \psi) \in [0, T] \times C$*, as in Pavel([11,Remark 5.3]). We denote* $\overline{D(A(t, \psi))} = \bar{D}$ *The dissipativity of* $A(t, \psi)$ *and the condition* $(A.4)$ *of Kartsatos and Parrott([5]) imply (H.2) when the domain of the single valued operator* $A(t, \psi)$ *is independent of* (t, ψ) *and* X^* *is uniformly convex. (See pavel[11].) Thus our results can be considered as a slight extenion of the results of Pavel([10],[11]) and Kartsatos and Parrott([5]).*

2. Main results

In the following disscussion, we assume that the hypotheses (H.1)–(H.4) hold. Let $u \in E$ be arbitrary but fixed. We first estabilsh the existence result of a limit solution for a type

$$\text{(FDE}:u) \qquad x'(t) \in A(t,u_t)x(t) + G(t,x_t), \quad t \in [0,T], \quad x_0 = \phi.$$

As in Tanaka[12], we have the following

LEMMA 2.1. *Let $\{t_j^n\}_{j=0}^n$ be a partition of the interval $[0,T]$, where $t_j^n = jh_n = j\frac{T}{n}$, $j = 0,1,\cdots,n$. If $n > (\omega+\beta)T$, then there exists $\{z_j^n\}_{j=0}^n$ in X such that for $j = 1,2,\cdots,n$,*

$$\frac{z_j^n - z_{j-1}^n}{h_n} \in A(t_j^n, u_{t_j^n})z_j^n + G(t_j^n, \bar{z}_j^n t_j^n), \tag{2.1}$$

where $z_0^n = \phi(0)$ and

$$\bar{z}_j^n(t) = \begin{cases} \phi(t), & t \in [-r,0] \\ z_k^n, & t \in (t_{k-1}^n, t_k^n] \quad \text{for} \quad k = 1,2,\cdots,j-1 \\ z_j^n, & t \in (t_{j-1}^n, T]. \end{cases}$$

The function $x_u^{(n)}(t) = \bar{z}_n^n(t)$ is said to be an approximate solution of (FDE:u).

LEMMA 2.2. *The sequences $\{z_j^n\}$ and $\{\frac{z_j^n - z_{j-1}^n}{h_n}\}$ is uniformly bounded.*

Proof. . Let $\omega_0 = \max(0,\omega)$ and $\bar{0}$ denotes the zero function on C. If $t \in (t_{j-1}^n, T]$, (1.3) and $\|u_t\|_C \le \|\phi\|_C + MT$ imply

$$\begin{aligned}
&(1-\omega_0 h_n)\|\bar{z}_j^n(t) - \phi(0)\| = (1-\omega_0 h_n)\|z_j^n - \phi(0)\| \\
&\le \|z_j^n - h_n(\frac{z_j^n - z_{j-1}^n}{h_n} - G(t_j^n, (\bar{z}_j^n)_{t_j^n})) - \phi(0)\| + h_n|A(0,\phi(0))\phi(0)| \\
&+ h_n|t_j^n - 0|L(\|u_{t_j^n}\|_C, \|\phi\|_C, \|\phi(0)\|)(1 + M + |A(0,\phi)\phi(0)|) \\
&\le \|z_{j-1}^n - \phi(0)\| + h_n\|G(t_j^n, \bar{z}_j^n) - G(t_j^n, \bar{0})\| \\
&+ h_n\|G(t_j^n, \bar{0})\| + h_n|A(0,\phi(0))\phi(0)| \\
&+ h_n TL(\|\phi\|_C + MT, \|\phi\|_C + MT, \|\phi(0)\|)(1 + M + |A(0,\phi)\phi(0))|) \\
&\le \|z_{j-1}^n - \phi(0)\| + h_n\beta\|(\bar{z}_j^n)_{t_j^n} - \phi(0)\|_C + h_n C_1
\end{aligned}$$

for $j = 1, 2, \cdots, n$, where

$$\begin{aligned} C_1 =& \beta\|\phi(0)\| + TL_1(0) + \|G(0,\tilde{0})\| + |A(0,\phi(0))\phi(0)| \\ &+ TL(\|\phi\|_C + MT, \|\phi(0)\|_C + MT, \|\phi(0)\|)(1 + M + |A(0,\phi)\phi(0)|) \end{aligned}$$

and

$$\bar{z}_0^n(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0), & t \in [0, T]. \end{cases}$$

If $t \in [-r, t_{j-1}^n]$, then

$$\begin{aligned} (1 - \omega_0 h_n)\|\bar{z}_j^n(t) - \phi(0)\| \le \|\bar{z}_j^n(t) - \phi(0)\| &= \|\bar{z}_{j-1}^n(t) - \phi(0)\| \\ &\le \|\bar{z}_{j-1}^n - \phi(0)\|_T, \end{aligned}$$

where $\|\psi\|_T = \sup_{t\in[-r,T]}\|\psi(t)\|$. Thus we have

$$(1 - \omega_0 h_n)\|\bar{z}_j^n - \phi(0)\|_T \le \|\bar{z}_{j-1}^n - \phi(0)\|_T + h_n\beta\|\bar{z}_j^n - \phi(0)\|_T + h_n C_1$$

which yields

$$[1 - (\omega_0 + \beta)h_n]\|\bar{z}_j^n - \phi(0)\|_T \le \|\bar{z}_{j-1}^n - \phi(0)\|_T + h_n C_1,$$

for $j = 1, 2, \cdots, n$. Applying the above inequality, we get

$$\|\bar{z}_n^n - \phi(0)\|_T \le \frac{2\|\phi\|_C + TC_1}{[1 - (\omega_0 + \beta)h_n]^n}.$$

If we take $(\omega_0 + \beta)h_n < \frac{1}{2}$, $(1 - (\omega_0 + \beta)h_n)^{-n} \le e^{2(\omega_0+\beta)T}$ since $(1 - h)^{-1} \le e^{2h}$ for $h \in [0, \frac{1}{2})$. Consequently, there exsits a constant $M_0 = M_0(\phi, T)$ such that

$$\sup_{n > 2(\omega_0+\beta)T}\{\max_{0\le j\le n}\|z_j^n\|\} \le M_0,$$

where $M_0 = e^{3(\omega_0+\beta)T}(3\|\phi\|_C + C_1 T)$.

Now we have a bound for $\{\frac{z_j^n - z_{j-1}^n}{h_n}\}$ with similar steps. Precisely,we note that from (1 3), we have

$$\frac{\|z_1^n - z_0^n\|}{h_n} \le e^{2\omega_0 h_n}(\beta M_0 + C_1)$$

From (1.1),

$$
\begin{aligned}
&(1-\omega_0 h_n)\|z_j^n - z_{j-1}^n\| \\
&\leq \|z_{j-1}^n - z_{j-2}^n + h_n(G(t_j^n,(\bar{z}_j^n)_{t_j^n}) - G(t_{j-1}^n,(\bar{z}_{j-1}^n)_{t_{j-1}^n}))\| \\
&\quad + {h_n}^2 L(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0) \\
&\quad \cdot (1 + M + \|\frac{z_{j-1}^n - z_{j-2}^n}{h_n}\| + \|G(t_{j-1}^n,(\bar{z}_{j-1}^n)_{t_{j-1}^n})\|) \\
&\leq (1 + h_n L(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0))\|z_{j-1}^n - z_{j-2}^n\| \\
&\quad + {h_n}^2[L_1(M_0) + L(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0) \\
&\quad \cdot (1 + M + \beta M_0 + L_1(0)T + \|G(0,\bar{0})\|)] \\
&\quad + h_n\beta(\max_{1\leq k\leq j}\|z_k^n - z_{k-1}^n\| + L_0 h_n)
\end{aligned}
$$

Here we used the facts that

$$
\|G(t_{j-1}^n,(\bar{z}_{j-1}^n)_{t_{j-1}^n})\| \leq \beta M_0 + L_1(0)T + \|G(0,\bar{0})\|
$$

and

$$
\begin{aligned}
\|(\bar{z}_j^n)_{t_j^n} - (\bar{z}_{j-1}^n)_{t_{j-1}^n}\|_C &= \sup_{\theta\in[-r,0]}\|\bar{z}_j^n(t_j^n+\theta) - \bar{z}_{j-1}^n(t_{j-1}^n+\theta)\| \\
&\leq \max_{1\leq k\leq j}\|z_k^n - z_{k-1}^n\| + L_0 h_n
\end{aligned}
$$

(See Kartsatos and Parrott[6]). Thus we have

$$
\begin{aligned}
&[1-(\omega_0+\beta)h_n]\frac{1}{h_n}\max_{1\leq k\leq j}\|z_k^n - z_{k-1}^n\| \\
&\quad \leq \frac{1}{h_n}\max_{1\leq k\leq j-1}\|z_k^n - z_{k-1}^n\|(1+h_n C_2) + h_n C_2,
\end{aligned}
$$

where

$$
\begin{aligned}
C_2 =& \beta L_0 + L_1(M_0) + L(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0) \\
&\cdot (1 + M + \beta M_0 + L_1(0)T + \|G(0,\bar{0})\|)
\end{aligned}
$$

Setting $p_n = 1-(\beta+\omega_0)h_n \in (0,\frac{1}{2})$, The above inequality implies

$$
\begin{aligned}
&\frac{p_n}{h_n}\max_{1\leq k\leq j}\|z_k^n - z_{k-1}^n\| \\
&\quad \leq h_n C_2 \sum_{s=0}^{j-2}\left[\frac{1+h_n C_2}{p_n}\right]^s + \frac{(1+h_n C_2)^{j-1}}{{p_n}^{j-2}}\cdot\frac{\|z_1^n - z_0^n\|}{h_n}
\end{aligned}
$$

for $j = 2, 3, \cdot, n$. It follows that

$$\begin{aligned}
&\frac{1}{h_n} \max_{1 \le k \le j} \|z_j^n - z_{j-1}^n\| \\
&\quad \le h_n C_2 \sum_{s=1}^{n} \left[\frac{1 + h_n C_2}{p_n} \right]^s + \left[\frac{1 + h_n C_2}{p_n} \right]^n (\beta M_0 + C_1) e^{2\omega_0 h_n} \\
&\quad = \left[\frac{1 + h_n C_2}{p_n} \right]^n (C_2 T + \beta M_0 + C_1) e^{2\omega_0 T}.
\end{aligned}$$

Since $(\frac{1+h_n C_2}{p_n})^n \le e^{(C_2 + 2\omega_0 + 2\beta)T}$, we conclude that there exist a constant M_1 such that

$$\sup_{n > 2(\omega_0 + \beta)T} \{ \frac{1}{h_n} \max_{1 \le k \le n-1} \|z_k^n - z_{k-1}^n\| \} \le M_1,$$

where

$$M_1 = e^{(6\omega_0 + 4\beta + C_2)T} [4\beta \|\phi\|_C + \|G(0, \bar{0})\| + |A(0, \phi)\phi(0)| + C_3 T]$$

and

$$\begin{aligned}
C_3 =& \beta C_1 + C_2 + L_1(0) \\
&+ (1 + M + |A(0, \phi)\phi(0)|) L(\|\phi\|_C + MT, \|\phi\|_C + MT, \|\phi(0)\|).
\end{aligned}$$

LEMMA 2.3. *Let $x_u^{(n)}(t)$ be an approximate solution of $(FDE : u)$ with $n > 2(\beta + \omega_0)T$. Then there exists a constant $M_2 = M_2(\phi, T)$ such that for $t, s \in [-r, T]$*

$$\|x_u^{(n)}(t) - x_u^{(n)}(s)\| \le M_2 |t - s| + 2M_2 h_n. \tag{2.3}$$

proof. We define the function

$$z_n(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ z_{j-1}^n + (t - t_{j-1}^n)(z_j^n - z_{j-1}^n)/h_n, & t \in (t_{j-1}^n, t_j^n], j = 1, 2, \cdots, n. \end{cases}$$

Then $z_n(t)$ is Lipschitz continuous with constant $M_2 = \max\{M_1, L_0\}$. Moreover, since

$$\begin{aligned}\|x_u^{(n)}(t) - z_n(t)\| &= \|z_j^n - z_{j-1}^n - (t - t_{j-1}^n)(z_j^n - z_{j-1}^n)/h_n\| \\ &= \|(h_n - (t - t_{j-1}^n))(z_j^n - z_{j-1}^n)/h_n\| \\ &= (t_j^n - t)\|(z_j^n - z_{j-1}^n)/h_n\| \le h_n M_2\end{aligned}$$

for $t \in (t_{j-1}^n, t_j^n]$,

$$\begin{aligned}&\|x_u^{(n)}(t) - x_u^{(n)}(s)\| \\ \le&\|x_u^{(n)}(t) - z_n(t)\| + \|z_n(t) - z_n(s)\| + \|z_n(s) - x_u^{(n)}(s)\| \\ \le&2M_2 h_n + M_2|t - s|\end{aligned}$$

for $t, s \in [-r, T]$.

The following gives an important estimate for the difference between the approximate solutions.

LEMMA 2.4. . *Let $\{z_j^n\}_{j=0}^n$ and $\{z_k^m\}_{k=0}^m$ be so constructed in Lemma2.1 and let $x_u^{(n)}(t)$ and $x_u^{(m)}(t)$ be the corresponding approximate solutions of $(FDE : u)$. Then there exist constants $C_4 = C_4(\phi, T)$ and $\{\epsilon_{n,m}\}_{n,m \in N}$ with $\lim_{n,m\to\infty} \epsilon_{n,m} = 0 (\epsilon_{n,m} > 0)$ such that*
(2.4)
$$(1 - \omega_0 h_n)^j (1 - \omega_0 h_m)^k \|z_j^n - z_k^m\| \le C_4 C_{j,k} + E_j + j h_n (C_4 C_{j,k} + \epsilon_{n,m}).$$

for $j = 0, 1, \cdots, n$ and $k = 0, 1, \cdots, m$, where

$$C_{j,k} = (t_j^n - t_k^m)^2 + h_n t_k^n + h_m t_k^m{}^{\frac{1}{2}},$$

and

$$E_j = \beta \sum_{l=1}^{j} \sup_{t \in [-r, t_l^n]} \|x_u^{(n)}(t) - x_u^{(m)}(t)\| h_n$$

Proof. . From (1.1), we have

$$
\begin{aligned}
&(1-\frac{h_n h_m}{h_n+h_m}\omega_0)\|z_j^n - z_k^m\| \\
&\le \|z_j^n - z_k^m - \frac{h_n h_m}{h_n+h_m}\Big(\frac{z_j^n - z_{j-1}^n}{h_n} - G(t_j^n, (\bar{z}_j^n)_{t_j^n}) \\
&\quad - \frac{z_k^m - z_{k-1}^m}{h_m} + G(t_k^m, (\bar{z}_k^m)_{t_k^m}\Big)\| \\
&\quad + \frac{h_n h_m}{h_n+h_m} L(\|u_{t_j^n}\|_C, \|u_{t_k^m}\|_C, \|z_k^m\|) \\
&\quad \cdot |t_j^n - t_k^m|(1 + M + \|\frac{z_k^m - z_{k-1}^m}{h_n}\| + \|G(t_k^m, (\bar{z}_k^m)_{t_k^m})\|),
\end{aligned}
$$

We set $r_{j,k} = (1-\omega_0 h_n)^j (1-\omega_0 h_m)^k$ for $k = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. Since $r_{j,k} \le \max(1-\omega_0 h_n, 1-\omega_0 h_m) \le 1 - \frac{h_n h_m}{h_n+h_m}\omega_0$, we obtain

$$
\begin{aligned}
&r_{j,k}\|z_j^n - z_k^m\| \\
&\le \frac{h_n}{h_n+h_m} r_{j,k-1}\|z_j^n - z_{k-1}^m\| + \frac{h_m}{h_n+h_m} r_{j-1,k}\|z_{j-1}^n - z_k^m\| \\
&\quad + \frac{h_n h_m}{h_n+h_m}[(1 + M + M_1 + \beta M_0 + L_1(0)T + \|G(0,\bar{0})\|) \\
&\quad \cdot L(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0)|t_j^n - t_k^m| \\
&\quad + \|G(t_j^n, (\bar{z}_j^n)_{t_j^n}) - G(t_k^m, (\bar{z}_k^m)_{t_k^m})\|
\end{aligned}
\tag{2.5}
$$

By (H.3),(2.3), and lemma 2.1,

$$
\begin{aligned}
&\|G(t_j^n, \bar{z}_j^n, t_j^n) - G(t_k^m, \bar{z}_k^m . t_k^m)\| \\
&\le \beta \sup_{t\in[-r,t_j^n]} \|x_u^{(n)}(t) - x_u^{(m)}(t)\| + \beta M_2|t_j^n - t_k^m| + 2\beta M_2 h_m \\
&\quad + L_1(M_0)|t_j^n - t_k^m|,
\end{aligned}
$$

and note that

$$
|t_j^n - t_k^m| \le |t_{j-1}^n - t_k^m| + h_n \le C_{j-1,k} + h_n.
$$

Set $A_{j,k} = r_{j,k}\|z_j^n - z_k^m\|$. By the above estimates and (2.5) imply (2.6)

$$A_{j,k} \leq \frac{h_n}{h_n + h_m} A_{j,k-1} + \frac{h_m}{h_n + h_m} A_{j-1,k} + \frac{h_n h_m}{h_n + h_m} (\beta \sup_{t \in [-r, t_j^n]} \|x_u^{(n)}(t) - x_u^{(m)}(t)\| + C_5 C_{j-1,k} + \epsilon_{n,m}).$$

where

$$C_5 = (1 + M + M_1 + \beta M_0 + L_1(0)T + \|G(0,\bar{0})\|) \cdot L(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0) + \beta M_2 + L_1(M_0)$$

and $\epsilon_{n,m} = C_5 h_n + 2\beta M_2 h_m$. In the proof of Lemma 2.1, since

$$(1 - \omega_0 h_n)\|z_j^n - \phi(0)\| \leq \|z_{j-1}^n - \phi(0)\| + h_n(\beta M_0 + C_1),$$

for $j = 1, 2, \cdots, n$, we get

$$(1-\omega_0 h_n)^j \|z_j^n - \phi(0)\| \leq h_n(\beta M_0 + C_1) \sum_{s=0}^{j-1} (1-\omega_0 h_n)^s \leq C_{j,0}(\beta M_0 + C_1).$$

Therefore, the inequality (2.4) is shown to be true for $j = 0, 1, \cdots, n$ and$k = 0$, where $C_4 = \max\{\beta M_0 + C_1, C_5\}$ In the same way, we can show the inequality (2.4) holds for $j = 0$ and $k = 0, 1, \cdots, m$. Now let $1 \leq k \leq m$ and $1 \leq j \leq n$, and suppose that (2.4) holds for the pairs $(j-1, k)$ and $(j, k-1)$. Since

$$\frac{h_n}{h_n + h_m} C_{j,k-1} + \frac{h_m}{h_n + h_m} C_{j-1,k} \leq C_{j,k},$$

(2.6) implies (2.4)

We now prove a convergence theorem for differenceapproximations relative to $(FDE : u)$.

THEOREM 2.5. *There exists a limit solution $x_u(t)$ of $(FDE : u)$ on $[-r, T]$ for fixed $u \in E$. Moreover, $x_u(t)$ is Lipschitz continuous with Lipschitz constant M_2 on $[-r, T]$.*

Proof. We assume that n and m are sufficiently large so that $n, m > 2(\omega_0 + \beta)T$.Let $\{t_j^n\}_{j=0}^n$ and $\{t_k^m\}_{k=0}^m$ be two arbitrary partitions of

$[0,T]$, where $t_j^n = jh_n = j\frac{T}{n}$, $j = 0, 1, \cdots, n$, $t_j^n = kh_m = k\frac{T}{m}$, $k = 0, 1, \cdots, m$. Let $t \in (t_{j-1}^n, t_j^n] \cap (t_{k-1}^m, t_k^m]$. By the virtue of lemma 2.4, we have

$$\begin{aligned} \|x_u^{(n)}(t) - x_u^{(m)}(t)\| &= \|z_j^n - z_k^m\| \\ &\leq e^{4\omega T}[C_4\{(h_n + h_m)^2 + (h_n + h_m)T\}^{\frac{1}{2}} \\ &+ \beta \sum_{l=1}^{j} \sup_{t\in[-r,t_l^n]} \|x_u^{(n)}(t) - x_u^{(m)}(t)\| h_n \\ &+ T\{C_4\{(h_n + h_m)^2 + (h_n + h_m)T\}^{\frac{1}{2}} + \epsilon_{n,m}\}] \end{aligned} \tag{2.7}$$

since $-h_m \leq |t_j^n - t_k^m| = (t_j^n - t) - (t_k^m - t) \leq h_n$. We now define the functions

$$F_{n,m}(\tau) = \begin{cases} 0, & \tau = 0, \\ \sup_{s\in[-r,t_l^n]} \|x_u^{(n)}(s) - x_u^{(m)}(s)\|, \tau \in (t_{l-1}^n, t_l^n], l = 1, \cdots, n. \end{cases}$$

Then we obtain that

$$F_{n,m}(\tau) \leq \sup_{s\in[-r,\tau]} \|x_u^{(n)}(s) - x_u^{(m)}(s)\| + 4M_2 h_n + 2M_2 h_m \tag{2.8}$$

for $\tau \in [0, T]$. Indeed, let $\tau \in (t_{l-1}^n, t_l^n]$, note that

$$F_{n,m}(\tau) = \max\{ \sup_{s\in[-r,\tau]} \|x_u^{(n)}(s) - x_u^{(m)}(s)\|, \sup_{s\in[\tau,t_l^n]} \|x_u^{(n)}(s) - x_u^{(m)}(s)\|\}. \tag{2.9}$$

Let $s \in [\tau, t_l^n]$. By (2.3), we get

$$\begin{aligned} &\|x_u^{(n)}(s) - x_u^{m)}(s)\| \\ &\leq \|x_u^{(n)}(\tau) - x_u^{(m)}(\tau)\| + \|x_u^{(n)}(s) - x_u^{(n)}(\tau)\| + \|x_u^{(m)}(s) - x_u^{(m)}(\tau)\| \\ &\leq \sup_{s\in[-r,\tau]} \|x_u^{(n)}(s) - x_u^{(m)}(s)\| + 2M_2|s - \tau| + 2M_2(h_n + h_m) \\ &\leq \sup_{s\in[-r,\tau]} \|x_u^{(n)}(s) - x_u^{(m)}(s)\| + 4M_2 h_n + 2M_2 h_m. \end{aligned}$$

Combining this with (2.9),we have(2.8). Now, by (2.8),

$$\begin{aligned}
&\beta \sum_{l=1}^{j} \sup_{s\in[-r,t_l^n]} \|x_u^{(n)}(s) - x_u^{(m)}(s)\|(t_l^n - t_{l-1}^n) \\
&= \beta \sum_{l=1}^{j} \int_{t_{j-1}^n}^{t_j^n} F_{n,m}(\tau)d\tau \le \beta \int_0^t F_{n,m}(\tau)d\tau + 2M_0\beta h_n \\
&\le \beta \int_0^t \sup_{s\in[-r,\tau]} \|x_u^{(n)}(s) - x_u^{(m)}(s)\|d\tau + (4M_2h_n + 2M_2h_m)\beta T + 2M_0\beta
\end{aligned}$$

From(2.7) and the above inequality,

$$\|x_u^{(n)}(t) - x_u^{(m)}(t)\| \le \delta_{n,m} + e^{4\omega_0 T}\beta \int_0^t \sup_{s\in[-r,\tau]} \|x_u^{(n)}(s) - x_u^{(m)}(s)\|d\tau,$$

where

$$\begin{aligned}
\delta_{n,m} =& e^{4\omega_0 T}[C_4\{(h_n + h_m)^2 + (h_n + h_m)T\}^{\frac{1}{2}} \\
&+ T\{C_4(h_n + h_m)^2 + (h_n + h_m)T\}^{\frac{1}{2}} + \epsilon_{n,m}\} \\
&+ (4M_2h_n + 2M_2h_m)\beta T + 2M_0\beta h_n].
\end{aligned}$$

We actually get that

$$\sup_{s\in[-r,t]} \|x_u^{(n)}(s) - x_u^{(m)}(s)\| \le \delta_{n,m} + \beta e^{4\omega_0 T} \int_0^t \sup_{s\in[-r,\tau]} \|x_u^{(n)}(s) - x_u^{(m)}(s)\|d\tau$$

for $t \in [0,T]$ which implies that $x_u^{(n)}(t)$ converges to a function $x_u(t)$ uniformly on $[-r,T]$. Finally, by (2.3), $x_u(t)$ is Lipschitz continuous on $[-r,T]$ with Lipschitz constant M_2.

Now we show the relation between the limit solutions of $(FDE:u)$ and $(FDE:v)$ for $u,v \in E$. From this result, we finally have the existence of a limit solution for (FDE).

THEOREM 2 6. *Let $x_u(t)$ and $x_v(t)$ be the limit solutions of $(FDE : u)$ and $(FDE : v)$, respectively. Then*

(2.10)

$$\|x_u(\bar{t}) - x_v(\bar{t})\|$$

$$\leq e^{(4\omega_0 T)}\Big(\|x_u(t) - x_v(t)\| + TL(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0)\|u - v\|_T$$

$$+ \int_t^{\bar{t}} < G(\eta, (x_u)_\eta) - G(\eta, (x_v)_\eta), x_u(\eta) - x_v(\eta) >_+ d\eta \Big)$$

for $0 \leq t \leq \bar{t} \leq T$

Proof. Let $\bar{x}_n^n(t)$ be an approximate solution for $x_u(t)$ such that

$$\frac{x_j^n - x_{j-1}^n}{t_j^n - t_{j-1}^n} \in A(t_j^n, u_{t_j^n})x_j^n + G(t_j^n, (\bar{x}_j^n)_{t_j^n}), \quad j = 1, 2, \cdots, n$$

$x_0 = \phi$ on $[-r, 0]$, $t_0^n = 0$, $x_0^n = \phi(0)$, $h_n = t_j^n - t_{j-1}^n$, $(\omega_0 + \beta)h_n < \frac{1}{2}$,

$$\bar{x}_j^n(t) = \begin{cases} \phi(0), & t \in [-r, 0], \\ x_l^n, & t \in (t_{l-1}^n, t_l^n], \quad l = 1, \cdots, j-1, \\ x_j^n, & t \in (t_{j-1}^n, T], \end{cases}$$

and $\lim_{n\to\infty} \bar{x}_n^n(t) = x_u(t)$ on $[-r, T]$. Also, we have an approximate solution $\bar{z}_m^m(t)$ for $x_v(t)$ such that

$$\frac{z_k^m - z_{k-1}^m}{t_k^m - t_{k-1}^m} \in A(t_k^m, v_{t_k^m})z_k^m + G(t_k^m, (\bar{z}_k^m)_{t_k^m}),$$

$k = 1, 2, \cdots, m$, $x_0 = \phi$ on $[-r, 0]$, $t_0^m = 0$, $x_0^m = \phi(0)$, $(\omega_0 + \beta)h_m < \frac{1}{2}$, $h_m = t_k^m - t_{k-1}^m$,

$$\bar{z}_k^m(t) = \begin{cases} \phi(0), & t \in [-r, 0], \\ z_l^m, & t \in (t_{l-1}^m, t_l^m], \quad l = 1, 2, \cdots, k-1, \\ z_k^m, & t \in (t_{k-1}^m, T], \end{cases}$$

and $\lim_{m\to\infty} \bar{z}_m^m(t) = x_v(t)$ on $[-r, T]$. Let $\delta \in (0, \frac{T}{2})$. and assume that n and m are sufficiently large so that $\max\{h_n, h_m\} < \delta$. Then there is a constant C_6 such that

$$
\begin{aligned}
&(1-\omega_0 h_n)^{j-p}(1-\omega_0 h_m)^{k-q}\|x_j^n - z_k^m\| \\
&\le \|x_p^n - z_q^m\| + C_6 C_{j,k} + \sum_{i=p}^{j} \delta_i^n h_n + \sum_{i=q}^{k} \delta_k^m h_m \\
&+ j h_n \{(\delta^{-1}\rho(T) + C_6)(C_{j,k} + |t_p^n - t_q^m|) + \rho(2\delta) + C_6 h_n \\
&+ L(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0)\|u - v\|_T\},
\end{aligned} \tag{2.11}
$$

for $j = p, \cdots, n$, $k = q, \cdots, m$, $p = 0, 1, \cdots, n$, $q = 0, 1, \cdots, m$, where

$$
\begin{aligned}
C_6 =&(1 + M + M_1 + 2\beta M_0 + 2TL_1(0) + 2\|G(0, \bar{0})\|) \\
&\cdot (1 + L(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0)).
\end{aligned}
$$

Here the symmbols used above are defined by

$$
\begin{aligned}
C_{j,k} = &\{((t_j^n - t_p^n) - (t_k^m - t_q^m))^2 + (t_j^n - t_p^n)h_n + (t_k^m - t_q^m)h_m\}^{\frac{1}{2}} \\
&+ \{((t_j^n - t_p^n) - (t_k^m - t_q^m))^2 + (t_j^n - t_p^n)h_n + (t_k^m - t_q^m)h_m\},
\end{aligned}
$$

$$
\delta_j^n = < G(t_j^n, (\bar{x}_j^n)_{t_j^n}) - G(t_j^n, (x_v)_{t_j^n}), x_j^n - x_v(t_j^n) >_\lambda,
$$

$$
\delta_k^m = \|G(t_k^m, (\bar{z}_k^m)_{t_k^m}) - G(t_k^m, (x_v)_{t_k^m})\| + \frac{2}{\lambda}\|z_k^m - x_v(t_k^m)\|,
$$

$$
\rho(t) = \sup_{|\tau - r| \le t} \{\|G(r, (x_v)_r) - G(\tau, (x_v)_\tau)\| + \frac{2}{\lambda}\|x_v(\tau) - x_v(r)\|\}
$$

and $< y, x >_\lambda = \lambda^{-1}(\|x + \lambda y\| - \|x\|)$, for $\lambda > 0$. In fact, we let $\sigma = h_n h_m/(h_n + h_m)$ and choose $0 < \lambda < 1$. Since

$$
\begin{aligned}
&x_j^n - z_k^m - \sigma\lambda\left(\frac{x_j^n - x_{j-1}^n}{h_n} - G(t_j^n, (\bar{x}_j^n)_{t_j^n}) - \frac{z_k^m - z_{k-1}^m}{h_m} + G(t_k^m, (\bar{z}_k^m)_{t_k^m})\right) \\
&= (1-\lambda)(x_j^n - z_k^m) + \frac{\lambda h_n}{h_n + h_m}(x_j^n - z_{k-1}^m) + \frac{\lambda h_m}{h_n + h_m}(x_{j-1}^n - z_k^m) \\
&+ \sigma\lambda\left(G(t_j^n, (\bar{x}_j^n)_{t_j^n}) - G(t_k^m, (\bar{z}_k^m)_{t_k^m})\right),
\end{aligned}
$$

(1.1) implies

$$
\begin{aligned}
&(1-\omega_0\sigma\lambda)(1-\lambda)\|x_j^n - z_k^m\| + (1-\omega_0\sigma\lambda)\lambda\|x_j^n - z_k^m\| \\
&\quad = (1-\omega_0\sigma\lambda)\|x_j^n - z_k^m\| \\
&\quad \le \|(1-\lambda)(x_j^n - z_k^m) + \sigma\lambda\Big(G(t_j^n,(\bar{x}_j^n)_{t_j^n}) - G(t_k^m,(\bar{z}_k^m)_{t_k^m})\Big)\| \\
&\quad + \frac{\lambda h_n}{h_n+h_m}\|x_j^n - z_{k-1}^m\| + \frac{\lambda h_m}{h_n+h_m}\|x_{j-1}^n - z_k^m\| \\
&\quad \sigma\lambda L(\|u_{t_j^n}\|_C, \|v_{t_k^m}\|_C, \|z_k^m\|)[|t_j^n - t_k^m|(1+|A(t_k^m, v_{t_k^m})z_k^m|) + \|u_{t_j^n} - v_{t_k^m}\|_C
\end{aligned}
$$

and then dividing this by λ, we have

(2.12)

$$
\begin{aligned}
&(1-\omega_0\sigma)\|x_j^n - z_k^m\| \\
&\quad \le \frac{1}{\xi}(\|(x_j^n - z_k^m) + \sigma\xi(G(t_j^n,(\bar{x}_j^n)_{t_j^n}) - G(t_k^m,(\bar{z}_k^m)_{t_k^m})\| - \|x_j^n - z_k^m\|) \\
&\quad + \frac{h_n}{h_n+h_m}\|x_j^n - z_{k-1}^m\| + \frac{h_m}{h_n+h_m}\|x_{j-1}^n - z_k^m\| \\
&\quad + \sigma L(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0) \\
&\quad [|t_j^n - t_k^m|(1+|A(t_k^m, v_{t_k^m})z_k^m|) + \|u_{t_j^n} - v_{t_k^m}\|_C],
\end{aligned}
$$

where $\xi = \lambda/(1-\lambda)$. We set $A_{j,k} = r_{j,k}\|x_j^n - z_k^m\|$. Dividing (2.12) by $1-\omega_0\sigma$ and $\lambda \to 0+$ in (2.12), we obtain

(2.13)

$$
\begin{aligned}
A_{j,k} &\le \frac{h_n}{h_n+h_m}A_{j,k-1} + \frac{h_m}{h_n+h_m}A_{j-1,k} \\
&\quad + \frac{h_n h_m}{h_n+h_m} < G(t_j^n,(\bar{x}_j^n)_{t_j^n}) - G(t_k^m,(\bar{z}_k^m)_{t_k^m}), x_j^n - z_k^m >_+ \\
&\quad + L(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0)[(C_{j-1,k} + |t_p^n - t_q^m| + h_n) \\
&\quad \cdot (1 + M + M_1 + \beta M_0 + TL_1(0) + \|G(0,\bar{0})\|) + \|u-v\|_T]
\end{aligned}
$$

Here we used the fact that $r_{j,k} \le \max\{1-\omega_0 h_n, 1-\omega h_m\} \le 1-\omega_0\sigma$,

$$
\begin{aligned}
&\|u_{t_j^n} - v_{t_k^m}\|_C \\
&\quad \le \|u_{t_j^n} - u_{t_k^m}\|_C + \|u_{t_k^m} - v_{t_k^m}\|_C \\
&\quad \le M|t_j^n - t_k^m| + \|u-v\|_T \\
&\quad \le M(|(t_j^n - t_p^n) - (t_k^m - t_q^m) - h_n| + |t_p^n - t_q^m| + h_n) + \|u-v\|_T \\
&\quad \le M(C_{j-1,k} + |t_p^n - t_q^m| + h_n) + \|u-v\|_T,
\end{aligned}
$$

and

$$|A(t_k^m, v_{t_k^m})z_k^m| \le \|\frac{z_k^m - z_{k-1}^m}{h_m}\| + \|G(t_k^m, (\bar{z}_k^m)_{t_k^m})\|$$
$$\le M_1 + \beta M_0 + TL_1(0) + \|G(0,\bar{0})\|.$$

Noting that $< b, a >_+ \le < b, o >_\lambda \le < d, c >_\lambda + \|b - d\| + \frac{2}{\lambda}\|a - c\|$,

$$\begin{aligned}&<G(t_j^n, (\bar{x}_j^n)_{t_j^n}) - G(t_k^m, (\bar{z}_k^m)_{t_k^m}), x_j^n - z_k^m >_+\\ &\quad \le < G(t_j^n, (\bar{x}_j^n)_{t_k^m}) - G(t_k^m, (x_v)_{t_j^n}), x_j^n - x_v(t_j^n) >_\lambda\\ &\quad + \|G(t_k^m, (\bar{z}_k^m)_{t_k^m}) - G(t_k^m, (x_v)_{t_k^m})\| + \|G(t_k^m, (x_v)_{t_k^m}) - G(t_j^n, (x_v)_{t_j^n})\|\\ &\quad + \frac{2}{\lambda}\|z_k^m - x_v(t_k^m)\| + \frac{2}{\lambda}\|x_v(t_k^m) - x_v(t_j^n)\|.\end{aligned}$$

Since

$$\rho(|t_j^n - t_k^m|) \le \delta^{-1}\rho(T)(|t_j^n - t_k^m| - h_n) + \rho(2\delta)$$
$$\le \delta^{-1}\rho(T)(C_{j-1,k} + |t_p^n - t_q^m|) + \rho(2\delta),$$

(2.13) is written as follows
(2.14)

$$\begin{aligned}A_{j,k} \le &\frac{h_n}{h_n + h_m}A_{j,k-1} + \frac{h_m}{h_n + h_m}A_{j-1,k}\\ &\frac{h_n h_m}{h_n + h_m}\{\delta_j^n + \delta_k^m + (C_6 + \delta^{-1}\rho(T))(C_{j-1,k} + |t_p^n - t_q^m|)\\ &+ \rho(2\delta) + C_6 h_n + L(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0)\|u - v\|_T\}\end{aligned}$$

At this moment, by (1.3)

$$\begin{aligned}&(1 - \omega_0 h_n)\|x_j^n - x_p^n\|\\ &\le \|x_{j-1}^n - x_p^n\| + h_n(\beta M_0 + TL_1(0) + \|G(0,\bar{0})\|) + h_n|A(t_p^n, u_{t_p^n})x_p^n|\\ &\quad + L(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0)(1 + M + |A(t_p^n, u_{t_p^n})x_p^n|)|t_j^n - t_p^n|h\end{aligned}$$

which yields

$$\begin{aligned}&(1 - \omega_0 h_n)^{j-p}\|x_j^n - x_p^n\|\\ &\le \sum_{i=p+1}^{j} C_6 h_n + \sum_{i=p+1}^{j} C_6|t_i^n - t_p^n|h_n \le C_6(|t_j^n - t_p^n| + |t_j^n - t_p^n|^2)\\ &\le C_6 C_{j,k}\end{aligned}$$

Thus

$$(1-\omega_0 h_n)^{j-p}\|x_j^n - z_q^m\| \le \|x_p^n - z_q^m\| + (1-\omega_0 h_n)^{j-p}\|x_j^n - x_p^n\|$$
$$\le C_6 C_{j,k} + \|x_p^n - z_q^m\|$$

for $j = p, \cdots, n$. This implies that (2.11) holds for $p \le j \le n$ and $k = q$.In the same way,(2.11)holds for $j = p$ and $q \le k \le m$. Next,let $p+1 \le j \le n$ and $q+1 \le k \le m$and suppose that (2.11) holds for the pair $(j-1,k)$ and $(j,k-1)$. Since (2.14) and

$$\frac{h_n}{h_n+h_m}C_{j,k-1} + \frac{h_m}{h_n+h_m}C_{j-1,k} \le C_{j,k},$$

it turns out that (2.11) holds for the fair (j,k). Hence, we conclude (2.14) holds for $p \le j \le n$ and $q \le k \le m$. Let $t \in (t_{p-1}^n, t_p^n] \cap (t_{q-1}^m, t_q^m]$ and $\bar{t} \in (t_{k-1}^m, t_k^m] \cap (t_{j-1}^n, t_j^n]$. Letting $n, m \to \infty$ in (2.11),

$$\begin{aligned} &\|x_u(\bar{t}) - x_v(\bar{t})\| \\ &\le \exp(4\omega_0 T)[\|x_u(t) - x_v(t)\| + \lim_{n\to\infty}\sup\sum_{i=p}^{j}\delta_i^n h_n \\ &\quad + \lim_{n\to\infty}\sup\sum_{i=q}^{k}\delta_i^m h_m + T\rho(2\delta) \\ &\quad + TL(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0)\|u-v\|_T]. \end{aligned} \tag{2.15}$$

Since

$$\lim_{n\to\infty}\sigma_{i=p}^{j}\delta_i^n h_n = \int_t^{\bar{t}} < G(\eta,(x_u)_\eta) - G(\eta,(x_v)_\eta), x_u(\eta) - x_v(\eta) >_\lambda d\eta$$

and $\lim_{m\to\infty}\sigma_{i=q}^{k}\delta_i^m = 0$, let $\delta \downarrow 0$ in (2 15), we have

$$\begin{aligned} &\|x_u(\bar{t}) - x_v(\bar{t})\| \\ &\le \exp(4\omega_0 T)[\|x_u(t) - x_v(t)\| TL(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0)\|u-v\|_T \\ &+ \int_t^{\bar{t}} < G(\eta,(x_u)_\eta) - G(\eta,(x_v)_\eta), x_u(\eta) - x_v(\eta) >_\lambda d\eta]. \end{aligned}$$

By letting $\lambda \downarrow 0$ for the above inequality, we finally have the desired inequality.

THEORM 2.7. *There exists $T_1 \in (0, T]$ such that (FDE) has a limit solution 0n $[0, T_1]$.*

Proof. Let $T_1 \in (0, T]$ be sufficiently small so that

$$T_1 \exp[4\omega_0 T_1 + \beta T_1 \exp(4\omega_0 T_1)] L(\|\phi\|_C + MT_1, \|\phi\|_C + MT_1, M_0) < 1$$

and $M_1 \le M$.Then by (2.10)and(H.3),

$$\begin{aligned}
&\|x_u(t) - x_v(t)\| \\
&\quad \le e^{(4\omega_0 T)}[TL(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0)\|u - v\|_T \\
&\qquad + \int_0^t \|G(\eta, (x_u)_\eta) - G(\eta, (x_v)_\eta)\| d\eta], \\
&\quad \le e^{(4\omega_0 T)} \Big[TL(\|\phi\|_C + MT, \|\phi\|_C + MT, M_0)\|u - v\|_T \\
&\qquad + \beta \int_0^t \|(x_u)_\eta - (x_v)_\eta\|_C d\eta \Big],
\end{aligned}$$

for $t \in [0, T_1]$. Thus we have

$$\begin{aligned}
&\sup_{s \in [-r, t]} \|x_u(t) - x_v(t)\| \\
&\quad \le T_1 L(\|\phi\|_C + MT_1, \|\phi\|_C + MT_1, M_0) e^{(4\omega_0 T_1)} \|u - v\|_{T_1} \\
&\qquad + \beta e^{(4\omega_0 T_1)} \int_0^t \sup_{s \in [-r, \eta]} \|(x_u)(s) - (x_v)(s)\| d\eta
\end{aligned}$$

for each $t \in [0, T_1]$. Consequently, $\|x_u - x_v\|_{T_1} < \|u - v\|_{T_1}$.Moreover, the limit solution $x_u(t)$ of $(FDE : u)$ satisfies

$$\|x_u(t) - x_u(s)\| \le M_2 |t - s| \le M|t - s|$$

for $t, s \in [-r, T_1]$ and $u \in E$. Therefore $x_u \in E$ for all $u \in E$. If we define an operator $S : E \mapsto E$ by $u \mapsto x_u$, where $x_u(t)$ is the limit solution of $(FDE : u)$. then S is a strictly contraction on a complete metric space E. By the Banach fixed point theorem, there is a unique fixed point of S in E, say $x(t)$ for $t \in [-r, T_1]$. Then such $x(t)$ is a limit solution of (FDE) which is Lipschitz continuous with Lipschitz constant M.

REMARK. It is obvious from the proof of the above theorems that the interval $[0, T]$ can be replaced by $[\bar{T}, T]$. Then the solution $x(t)$ of(FDE)exists beyind $\bar{T}$. With this processing, we may have the existence of a maximal interval of existence of solutions of (FDE) on $[0, T]$. Furthermore, using the result of Theorem 2.6 we can obtain the uniqueness of the solutions of (FDE) with the concept of integral solution defined by Benilan as in Ha,Shin and Jin [3]

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