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ON THE PARAMETER SPACES FOR TOPOLOGICAL TYPE (1,1)

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1. Introduction

A Riemann surface is a one-dimensional complex analytic manifold. A Riemann surface S is of (topological) type (g, n) if S is conformally a compact surface of genus q with n conformal discs removed.

This paper is an exploration of the space of Riemann surfaces of topological type (1, 1) ; that is, the space of distinct conformal structures on a torus with one boundary component. A Fuchsian group is any discontinuous group contained in the hyperbolic group. The hyperbolic group is the group of operations which change z to $\dfrac{az+b}{cz+d}$ and which leaves the fundamental circle fixed.

A Fuchsian group is completely determined when we know its generators, since by combining them in all possible ways we obtain all transformations in the group. In the Maskit's paper [5], he gave an explicit parametrization of the space of standard generators of Fuchsian groups representing tori with one boundary component. Here, we assume S is a Riemann surface whose universal covering space is the upper half plane U.

Let G be the Fuchsian group acting on U in such a way that $U/G = S$. In this paper, using Maskit's parameter space we are going to find some relations between the two generators $A=\begin{pmatrix} \lambda sx & x \\ \lambda s & x \end{pmatrix}$ and $B=\begin{pmatrix} 0 & x \\ -s & x(1+s) \end{pmatrix}$ with λ=1 and to find the exact value of $cos\theta$, where θ is the appropriate angle of intersection of the geodesic α and

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β associated to A and B respectively. Moreover, we show α intersects β only once.

2. Main Results

In the Maskit's parmeter space

$$P = \{(x, s, \lambda) \in \boldsymbol{R}^3 | x > 1, s > 0, \lambda \geq 1\}$$

for the Fuchsian groups of topological type (1, 1), if $\lambda = 1$ then U/G is a torus with one puncture and the group G can be generated by transformations

$$A = \begin{pmatrix} \lambda sx & x \\ \lambda s & x \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & x \\ -s & x(1+s) \end{pmatrix}$$

An easy calculation shows the following lemma.

LEMMA 1. . *If r_A and r_B are the repelling fixed points of A and B respectively and a_A and a_B are the attracting fixed points of A and B respectively, then*

(a) $r_A = \frac{x(s-1)-\sqrt{x^2(s-1)^2+4sx}}{2s}, \quad a_A = \frac{x(s-1)+\sqrt{x^2(s-1)^2+4sx}}{2s}$
(b) $r_B = \frac{x(s+1)+\sqrt{x^2(s+1)^2-4sx}}{2s}, \quad a_B = \frac{x(s+1)-\sqrt{x^2(s+1)^2-4sx}}{2s}$
(c) $r_A < 0 < a_B < 1 < a_A < x < r_B$.

Proof. (a) From $A(z) = \dfrac{sxz + x}{sz + x} = z$, we have $sz^2 + (x - sx)z - x = 0$
Hence

$$z = \frac{x(s-1) \pm \sqrt{x^2(s-1)^2 + 4sx}}{2s}$$

Therefore

$$r_A = \frac{x(s-1) - \sqrt{x^2(s-1)^2 + 4sx}}{2s}$$

$$a_A = \frac{x(s-1) + \sqrt{x^2(s-1)^2 + 4sx}}{2s}.$$

(b) Similarly for (a).
(c) By easy calculation we can show that

$$r_A < 0 < a_B < 1 < a_A.$$

To show $a_A < x$, that is , $\frac{x(s-1)+\sqrt{x^2(s-1)^2+4sx}}{2s} < x$ we use the fact that $x > 1$, then it can also be derived. Similary we have $x < r_B$. Therefore, $r_A < 0 < a_B < 1 < a_A < x < r_B$.

THEOREM 2. *If α and β are the geodesics associated to A and B respectively and θ is the appropriate angle of intersection of α and β then*

$$\cos\theta = \frac{x(1-s^2)}{\sqrt{x^2(s^2-1)^2+16s^2(x-1)}}.$$

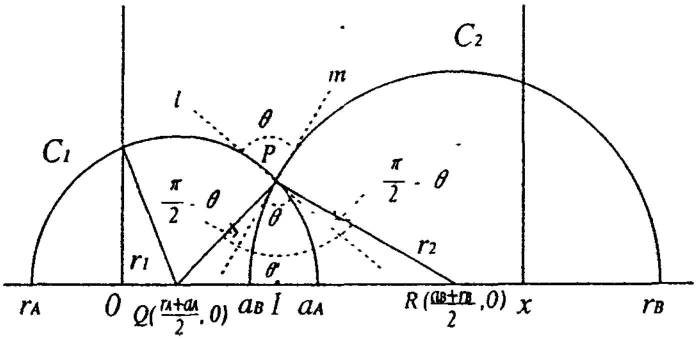
Proof. Draw two circles C_1 with center $Q(\frac{r_A+a_A}{2},0)$ and radius $\frac{a_A-r_A}{2}$, C_2 with center $R(\frac{a_B+r_B}{2},0)$ and radius $\frac{r_B-a_B}{2}$ and let P be the intersection point of two circles C_1, C_2 in the upper half-plane U.

At the point P, draw two tangent lines l and m with respect to the circles C_1 and C_2 respectively. Let θ be the angle between l and m as in the Figure 1. In the triangle PQR, if we let $\theta' = \angle QPR$ then we have $\theta' = (\frac{\pi}{2}-\theta)+\theta+(\frac{\pi}{2}-\theta) = \pi-\theta$. Hence $\cos\theta' = \cos(\pi-\theta) = -\cos\theta$. Since $\cos\theta' = \dfrac{\overline{PQ}^2+\overline{PR}^2-\overline{QR}^2}{2\overline{PQ}\cdot\overline{PR}}$ and

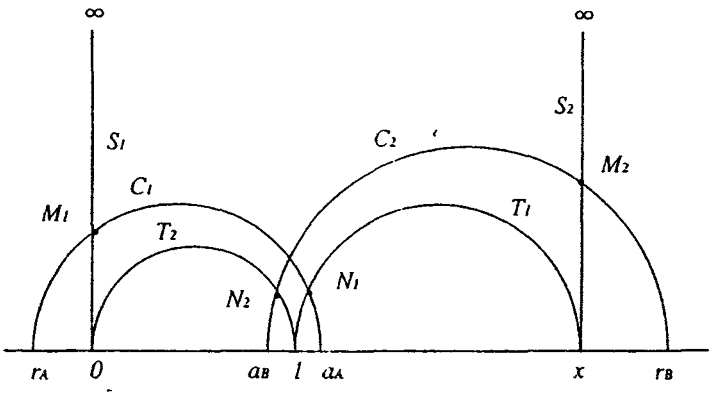
$$\overline{PQ} = \frac{a_A-r_A}{2} = \frac{\sqrt{x^2(s-1)^2+4sx}}{2s}$$
$$\overline{PR} = \frac{r_B-a_B}{2} = \frac{\sqrt{x^2(s+1)^2-4sx}}{2s}$$
$$\overline{QR} = \frac{a_B+r_B}{2} - \frac{r_A+a_A}{2}\frac{x(s+1)}{2s} - \frac{x(s-1)}{2s} = \frac{x}{s}.$$

We have

$$\cos\theta = \frac{x(1-s^2)}{\sqrt{x^2(s^2-1)^2+16s^2(x-1)}}.$$



(Figure 1)



(Figure 2)

A canonical fundamental polygon D for G is the region bounded by the four hyperbolic lines, S_1, T_1, S_2 and T_2 in the upper half-plane U. S_1 is the line with endpoints at 0 and ∞, T_1 is the line with endpoints at 1 and x, S_2 is the line with endpoints at x and ∞ and T_2 is the line with endpoints at 0 and 1.

THEOREM 3. . *Let α and β be as in the Theorem 2. Let the axis $\widehat{r_A a_A}$ of A intersects S_1 and T_1 at M_1 and N_1 respectively and the* *axis* $\widehat{a_B r_B}$ *of* B *intersects* S_2 *and* T_2 *at* M_2 *and* N_2 *respectively. Then* $A(M_1) = N_1$ *and* $B(M_2) = N_2$.

Proof. Let C_1 and C_2 be as in the Theorem 2. The hyperbolic line T_1 with endpoints 1 and x is a circle with center $(\frac{x+1}{2}, 0)$ and radius $\frac{x-1}{2}$, call it C_3. The hyperbolic line T_2 with endpoints 0 and 1 is a circle with center $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$, call it C_4. (see the Figure 2)

Then the coordinate for M_1 is $(0, \sqrt{\frac{x}{s}})$ since

$$\sqrt{(\frac{a_A - r_A}{2})^2 - (\frac{r_A + a_A}{2})^2} = \sqrt{\frac{x}{s}}.$$

The coordinate for N_1 is $(\frac{x+xs}{x+s}, \frac{x-1}{x+s} s\sqrt{\frac{x}{s}})$. For, let (X, Y) be the coordinate for N_1 then from the circles C_1 and C_3 ;

$$(X - \frac{x(s-1)}{2s})^2 + Y^2 \Big(\frac{\sqrt{x^2(s-1)^2 + 4sx}}{2s}\Big)^2$$

$$(X - \frac{x+1}{2})^2 + Y^2 = (\frac{x-1}{2})^2$$

We have $X = \frac{x+xs}{x+s}$ and $Y = \frac{x-1}{x+s} s\sqrt{\frac{x}{s}}$. e may wirte $z_1 = i\sqrt{\frac{x}{s}}$ for $M_1(0, \sqrt{\frac{x}{s}})$and $z_2 = \frac{x+xs}{x+s} + i\frac{x-1}{x+s} s\sqrt{\frac{x}{s}}$ for $N_1(\frac{x+xs}{x+s}, \frac{x-1}{x+s} s\sqrt{\frac{x}{s}})$. So

$$\begin{aligned} A(z_1) &= \frac{sxz_1 + x}{sz_1 + x} = \frac{sx(i\sqrt{\frac{x}{s}}) + x}{s(i\sqrt{\frac{x}{s}}) + x} = \frac{x + ix\sqrt{sx}}{x + i\sqrt{sx}} \\ &= \frac{x+sx}{x+s} + i\frac{x-1}{x+s} s\sqrt{\frac{x}{s}} \\ &= z_2. \end{aligned}$$

Thus $A(M_1) = N_1$.

Similarly, by considering the circles C_2 and C_4 we have that the coordinate for M_2 is $(x, \sqrt{\frac{x(x-1)}{s}})$, the coordinate for N_2 is $(\frac{x}{x+s(x-1)}, \frac{\sqrt{sx(x-1)}}{x+s(x-1)})$ and $B(M_2) = N_2$.

REMARK. *From Theorem 3, we know that* α *is* $\widehat{r_A a_A} \bigcap D$ *and* β *is* $\widehat{a_B r_B} \bigcap D$. *Hence* α *intersects* β *only once at* P. *Therefore using Theorem 2 with the help of Wolpert's cosine formula [6] and Gardiner formula [2], we can find a Fuchsian group of type* $(1,1)$ *whose Petersson inner product is 0 [4].*

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