Honam Mathematical J. 17(1995), No. 1,pp. 7–14

THE STRUCTURE OF A CONNECTED LIE GROUP G WITH ITS LIE ALGEBRA $\mathfrak{g} = rad(\mathfrak{g}) \oplus \mathfrak{sl}(2, \mathbb{F})$

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1. Abstract

The purpose of this study is to construct the structure of the connected Lie group G with its Lie algebra $\mathfrak{g} = rad(\mathfrak{g}) \oplus \mathfrak{sl}(2, \mathbb{F})$, which conforms to Stellmacher's [4] Pushing Up. The main idea of this paper comes from Stellmacher's [4] Pushing Up. Stellmacher considered Pushing Up under a finite p-group. This paper, however, considers Pushing Up under the connected Lie group G with its Lie algebra $\mathfrak{g} = rad(\mathfrak{g}) \oplus \mathfrak{sl}(2, \mathbb{F})$. In this paper, $O_p(G)$ in [4] is Q=exp($\mathfrak{q}$), where $\mathfrak{q} = nilrad(\mathfrak{g})$ and a Sylow p-subgroup S in [7] is S=exp($\mathfrak{s}$), where $\mathfrak{s} = \mathfrak{q} \oplus \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \mid * \in \mathbb{F} \right\}$.

Showing the properties of the connected Lie group and the subgroups of the connected Lie group with relations between a connected Lie group and its Lie algebras under the exponential map, this paper constructs the subgroup series $C_Z(G) < Z < Q < G$ and shows $[M, Q]$=1, where M is the maximal semisimple connected subgroup of G.

In this paper, we usually denote Lie algebras by lowercase German letters.

2. Main Hypothesis

Part I : Assume that G is a connected Lie group over $\mathbb{F}(= \mathbb{R}, \mathbb{C})$. Let $\mathfrak{g}$ is the Lie algebra of G. Assume that $\mathfrak{g} = rad(\mathfrak{g}) \oplus \mathfrak{sl}(2, \mathbb{F})$.

When, by Theorem 3.18.13 in [5], Levi-decomposition gives that $G = RM$,

where $R = rad(G)$ and M is the maximal semisimple connected subgroup of G.

Received March 5, 1995.

Part II: Assume that G is a subgroup of a group $H = \langle G, N_H(S)\rangle$, where H is generated by G and $N_H(S)$, and $N_H(S)$ is the normalizer of S in H. Also $N_H(S)$ induces a Lie group automorphisms of S.

We estabilish notations as follows.

Let $\mathfrak{s}_o$ is the subalgebra of $\mathfrak{sl}(2,\mathbb{F})$ given by $\left\{\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} | * \in \mathbb{F}\right\}$.

$\mathfrak{q} = nilrad(\mathfrak{g})$
$\mathfrak{s} = \mathfrak{q} \oplus \mathfrak{s}_o$
$S_o = \exp(\mathfrak{s}_o)$
$S = \exp\ (\mathfrak{s})$
$Q = \exp\ (\mathfrak{q})$
$\mathfrak{m}$= the Lie algebra of $M \cong \mathfrak{sl}(2,\mathbb{F})$.

Put $G_o = QM$.

Now, we want to replace G by G_o in main hypothesis.

LEMMA 2.1. *$G = N_G(S)\ G_o$, where $N_G(S)$ is the normalizer of S in G.*

Proof. We have that G/Q is reductive and that $R/Q \leq G/Q$ is the radical of G/Q.
However, the radical of a reductive group is in the center. Thus $R/Q \leq Z(G/Q)$ and so $[R,S] \leq Q \leq S$. Hence $R \leq N_G(S)$. We then have $G = RM \leq N_G(S)G_o$.
Therefore, $G = N_G(S)G_o$.

Next, we need to show that G_o is connected. Since M and Q are connected Lie groups, $G_o = MQ$ is a connected Lie group by Bourbaki in [1]. Now, we have that the Lie algebra of G_o is $\mathfrak{q} \oplus \mathfrak{m}$ and $G = N_G(S)\ G_o$ in Lemma 2.1. Thus $H = \langle G, N_H(S)\rangle = \langle N_G(S)G_o, N_H(S)\rangle = \langle G_o, N_H(S)\rangle$. Hence the main hypothesis is satisfied by G_o in place of G. Also, if G_o satisfies the conclusion of main theorem, then so dose G. Therefore we assume that $G = G_o$.
Also, $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{sl}(2,\mathbb{F})$ as $\exp(\mathfrak{g}) \geq \exp(\mathfrak{q} \oplus \mathfrak{m}) \geq \langle \exp(\mathfrak{q}), \exp\ (\mathfrak{m})\rangle = QM = G$.
Therefore we have,

PRPPOSITION 2.2. *$G = QM$ and $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{sl}(2,\mathbb{F})$.*

3. The Structure of $Z(Q)$

In this section, we will construct the structure of the center of Q.

We quote the following Lemma from proposition 3.26 in [6].

LEMMA 3.1. *Let G and H be connected Lie groups, and let $\phi : G \to H$ be a homomorphism. Then ϕ is a covering map if and only if $d\phi : G_e \to H_e$ is an isomorphism.*

LEMMA 3.2. *M is a covering group of $PSL(2, \mathbb{F})$.*

Proof. We know the M is a connected Lie group with a Lie algebra $\mathfrak{sl}(2, \mathbb{F})$ by the definition. Now, $\mathrm{SL}(2, \mathbb{F})$ is a covering group of $\mathrm{PSL}(2, \mathbb{F})$ as $\mathrm{SL}(2, \mathbb{F}) \to \mathrm{PSL}(2, \mathbb{F})$ is a natural homomorphism with discrete kernel $\{\pm I\}$ and is continuous, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence, the Lie algebas of $\mathrm{SL}(2, \mathbb{F})$ is isomorphic to the Lie algebra of $\mathrm{PSL}(2, \mathbb{F})$ by Lemma 3.1. Thus the Lie algebra of $\mathrm{PSL}(2, \mathbb{F})$ is $\mathfrak{sl}(2, \mathbb{F})$.

Let M^* denote the universial covering group of $\mathrm{PSL}(2, \mathbb{F})$. Then M^* is the unique (up to isomorphism) simply connected Lie group with its Lie algebra $\mathfrak{sl}(2, \mathbb{F})$. Now M is a connected Lie group with Lie algebra $\mathfrak{sl}(2, \mathbb{F})$. So M^* is the universial cover of M.

Thus we have the following diagram:

$$\begin{array}{ccc} M^* & = & M^* \\ {\scriptstyle covering}\downarrow & & \downarrow{\scriptstyle covering} \\ M^*/D = \mathrm{PSL}(2, \mathbb{F}) & \longleftarrow & M = M^*/D^* \end{array}$$

Here D^*, D discete normal subgroups of M^*.

We need to prove that $D^* \leq D$. If so, then
$\mathrm{PSL}(2, \mathbb{F}) = M^*/D \cong (M^*/D^*)/(D/D^*) = M \,/\, (D/D^*)$.
Since D/D^* is discrete normal in M, M is a covering group of $\mathrm{PSL}(2, \mathbb{F})$.
To prove the claim that $D^* \leq D$: note that DD^* is normal in M^*.
We have $M^*/DD^* \cong (M^*/D)/(DD^*/D) = \mathrm{PSL}(2, \mathbb{F})/(DD^*/D)$.
Since $\mathrm{PSL}(2, \mathbb{F})$ is simple, (DD^*/D) is either 1 or $\mathrm{PSL}(2, \mathbb{F})$.
If $DD^*/D = 1$, then $D^* \leq D$ as desired.
Suppose $\mathrm{PSL}(2, \mathbb{F}) = DD^*/D$. Then $\mathrm{PSL}(2, \mathbb{F}) = DD^*/D \cong D^*/(D^* \cap D)$ is discrete.
It is not the case.

Put $Z = Z(Q)$.
Then Z is a connected Lie subgroup of G by cor.3.6.4 in [5] and cor.3.50 (a) in [6].

LEMMA 3.3. *Let $\mathfrak{g}$ be an abelian Lie algebra and let G be an additive group of $\mathfrak{g}$.*

Then $\mathfrak{g}$ *is a Lie algebra of* G *and exp:* $\mathfrak{g} \to G$ *is an identity.*

Proof. Since G is an abelian vector space, G is a Lie group.
For any $a \in G$, let $\phi_a : F \to G$ be a 1-parameter subgroup of G by $t \mapsto ta$. Then let X_a be the unique invariant vector field on G whose value at identity e is $X_a(e) = d\phi_a(\frac{d}{dt}|_0)$.
By definition, this means that for any $c \in G$ and any $f \in \mathcal{C}^\infty(G)$, we have $X_a(c) = dl_c(X_a(e))$ where $l_c : G \to G$ is a translation by c, i.e., $l_c(b) = b + c$, and we thus have $X_a(c)(f) = \frac{d}{dt}|_0(f \circ l_c \circ \phi_a)$.

We want to show that $X_a(c)$ is the directional derivative at c, in the direction a.
Fix a coordinate function $(x_i)_{i=1}^n$ for G, and take $a = (a_1, \cdots, a_n)$.
Put $\zeta = l_c \circ \phi_a$.
Then the chain rule gives

$$X_a(c)(f) = \frac{d}{dt}|_0(f \circ \zeta) = \sum_{n=1}^{n} \frac{\partial f}{\partial x_i}(\zeta(0))\, \zeta_i'(0)$$

where $\zeta(t) = (\zeta_1(t), \cdots, \zeta_n(t)) = (c_i + ta_i)_{i=1}^n$.
Thus, $X_a(c)(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}(c)$, and so that $X_a(c)$ is the directional derivative at c, in the direction a.

We now show that any invariant vector field X on G is of the form X_a for some a.
Indeed, it is enough to show that $X(e) = X_a(e)$ for some a.
Now, simply take $a = (X(e)(x_1), \cdots, X(e)(x_n))$, where x_i is the i^{th} coordinate function $G \to \mathbb{F}$.
Then $X(e)$ and $X_a(e)$ agree on all polynomial maps f, and since these are dense in $\mathcal{C}^\infty(G)$, we get $X = X_a$.

Define $\lambda : \mathfrak{g} \to L(G)$ by $a \mapsto X_a$, where $L(G)$ is a Lie algebra of G. As we have just now shown, λ is surjective. Also, λ is linear. Indeed, we have $X_{a+b} = X_a + X_b$ and $X_{ca} = c\, X_a$. Evidently, the directional derivative $X_a(e)$ is zero if and only if $a = 0$, so λ is an isomophism of vector spaces. In order to show that λ is an isomorphism of Lie algebras, we need only observe that $L(G)$ is abelian; $([X,Y](e))\, f = 0$ for any $X = X_a$, $Y = X_b$. For this it suffices to let f be a coordinate function. The fact that $L(G)$ is abelian then reduces to the commutativity of multiplication in $\mathbb{F}$.

We now identify $\mathfrak{g}$ with $L(G)$ via λ. Notice that exp: $L(G) \to G$ is given by $X_a \mapsto \phi_a(1) = a$, so under the above identification, we have exp=id : $\mathfrak{g} \to G$.

PROPOSITION 3.4. *$Z = C_Z(G) \oplus W$, where W is a G-invariant subgroup of Z, $W \cong \mathbb{F}^n$ for some n and G acts linearly on W.*

Proof. We have that Z is a connected Lie group. Hence, Z has a simply connected universial covering group Z^*. Then Z^* is a simply connected abelian group by Corollary 3.50 (b) in [6] and so $Z^* \cong \mathbb{F}^n$ for some n. Let $\pi : Z^* \to Z$ be a covering map.
Then $L(Z^*) \cong L(Z)$ by Lemma 3.1. By Lemma 3.3, we have $L(Z^*) = Z^*$.

Now, we have a commutative diagram by 3.46 (6) in [6]:

$$(*) \qquad \begin{array}{ccc} G & \xrightarrow{\;\mathfrak{a}_\sigma\;} & G \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ L(G) & \xrightarrow[Ad_\sigma]{} & L(G) \end{array}$$

where $\mathfrak{a}_\sigma$ is an inner automorphism of G given by conjugation by σ.
Then $\exp|_{Z^*} = \pi$ is G - equivariant by the commutative diagram.
i.e., $\exp|_{Z^*}(z^*g) = \pi(z^*g) = g^{-1}(\pi(z^*))g = g^{-1}(\exp|_{Z^*}(z^*))g$, for $g \in G$ and $z^* \in Z^*$.

Now, set $\mathrm{Ker}(\pi) = D$.
Then we have D is a G-invariant.
Note that G acts linearly on Z^* via Ad, see (*).
Since the only continuous action of a connected Lie group on a discrete set is trivial, M centralizes D. Then Weyl's Theorem says that $Z^* = C_{Z^*}(M) \oplus W$ for some M-invariant subspace W of Z^*. Hence $D \leq C_{Z^*}(M)$.
So, $Z = Z^*/D = (C_{Z^*}(M) \oplus W)/D = (C_{Z^*}(M)+D)/D \oplus (W+D)/D = C_Z(M) \oplus W'$,
where $W' = (W+D)/D \cong W/(W \cap D) = W$ as $W \cap D = 1$.
Identify W with W'. Then $Z = C_Z(M) \oplus W$.
Since $G = QM$ and Q centralizes Z, we have $Z = C_Z(G) \oplus W$.

4. Main Theorem

DEFINITION 4.1. $O(G)$ denotes the subgroup of G generated by semisimple subgroups of G.

Then we have the following lemma:

LEMMA 4.2.

(1) *$G/O(G)$ is nilpotent.*

(2) *If N is a normal subgroup of G such that G/N is nilpotent, then* $N \geq O(G)$.

Proof. (1) By the definition of $O(G)$, $O(G)$ is generated by all M^x for $x \in G$ and hence $G = QO(G)$.
Thus $G/O(G) \cong Q$ is nilpotent.

(2) Suppose N is normal of G such that G/N is nilpotent. Assume that $N < O(G)$.
Then $G/N \geq O(G)N/N \cong O(G)/(N \cap O(G))$ is not nilpotent, a contradiction.
Thus $N \geq O(G)$.

LEMMA 4.3. *$S = QS_o$ and $S_o \cap Q = 1$.*

Proof. By the definition of S, $S = \exp(\mathfrak{s}) = \exp(\mathfrak{q} \oplus \mathfrak{s}_o)$.
Clearly, $QS_o \leq S$. Conversely, we have QS_o is a connected Lie group by Bourbaki in [1]. Let X be a corresponding Lie algebra of QS_o under exp: $\mathfrak{g} \to G$.
Then $S = \exp(\mathfrak{s}) = \exp(\mathfrak{q} \oplus \mathfrak{s}_o) \leq \exp(X) = QS_o$. Thus $S = QS_o$.

Now, we have a cimmutative diagram:

$$\begin{array}{ccc} \mathfrak{sl}(2,\mathbb{F}) & \xrightarrow{\exp_1} & M \\ \| & & \downarrow \pi \\ \mathfrak{sl}(2,\mathbb{F}) & \xrightarrow[\exp_2]{} & \mathrm{PSL}(2,\mathbb{F}) \end{array}$$

So, we have $\pi \circ \exp_1 = \exp_2$ and if $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in \mathfrak{s}_o$, then $exp_2(A)$ is the image of $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ in $PSL(2,\mathbb{F})$.

Hence we have $\pi(S_o) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} | * \in \mathbb{F} \right\}$ and so

$$\exp_2|_{\mathfrak{s}_o} : \mathfrak{s}_o \longrightarrow \pi(S_o)$$

is an isomorphism. By the commutative diagram, we have $\pi \circ \exp_1 |_{\mathfrak{s}_o}$ is injective. Since $\mathrm{PSL}(2,\mathbb{F})$ is simple, $M \cap Q \leq \operatorname{Ker} \pi$.
Thus, $S_o \cap Q \leq M \cap Q \leq \operatorname{Ker} \pi$.
Since $\pi|_{\exp(\mathfrak{s}_o)}$ is injective, $\pi|_{S_o}$ is injective. Hence we have $S_o \cap Q = 1$.

LEMMA 4.4. *$Z(S)$ is connected.*

Proof. Since $S = QS_o$, S is connected. Also, S is nilpotent as $\mathfrak{s}$ is nilpotent. Thus, by Corollary 3.6.4 in [5], $Z(S)$ is connected.

THEOREM 4.5. *Assume that Z and S are defined as above. Then we have*

$$(i)\ Z \leq \langle\, (Z(S))^G \,\rangle).$$
$$(ii)\ \text{If } Z(S) \leq Q, \text{ then } Z = \langle\, (Z(S))^G \,\rangle.$$

Proof. (i) By Proposition 3.4, we have $Z^* = C_{Z^*}(M) \oplus W$ and $C_{Z^*}(G) = Z(\mathfrak{g})$ by Bourbaki in [1]. By Weyl's Theorem, $W = W_1 \oplus \cdots\cdots \oplus W_t$, irreducible summands for the action of $\mathfrak{m}$. Since $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{m}$, each W_i is a $\mathfrak{g}$-submodule of Z^*. Since $\mathfrak{s}_o$ is an ad-nilpotent subgroup of $\mathfrak{m} \cong \mathfrak{sl}(2, \mathbb{F})$, $\mathfrak{s}_o$ acts as a Lie algebra of nilpotent linear transformations on each W_i. Hence $C_{W_i}(\mathfrak{s}_o) \neq 0$ and so $C_{W_i}(S_o) \neq 0$. Since $S = S_oQ$ by Lemma 4.3, $C_{W_i}(S) \neq 0$. Since W_i is irreducible, $W_i \langle\, (C_{W_i}(S)^G) \,\rangle$. Also, $C_{W_i}(S) \leq C_Z(S) \leq C_Q(S) \leq Z(S)$. Thus, we have $W_i \leq \langle (Z(S))^G \rangle$ and so $W = \langle (Z(S))^G \rangle$.
Since $C_Z(G) \leq C_Z(S) \leq C_Q(S) \leq Z(S)$ and $C_Z(G)$ is normal in G, $C_Z(G) \leq \langle (Z(S))^G \rangle$. Thus we have $Z \leq \langle (Z(G))^G \rangle$ from the form $Z = C_Z(G) \oplus W$ by Proposition 3.4.
(ii) If $Z(S) \leq Q$, then $Z(S) \leq Z(Q)$ as $Z(S) \leq Q \leq S$.
So, $\langle (Z(S))^G \rangle \leq Z(Q) = Z$ as $Z(Q)$ is normal in G.
Thus we have $Z = \langle (Z(S))^G \rangle$.

LEMMA 4.6. *Assume that M and Q are defined as above. Then $[M, Q]$ =1.*

Proof. We know that $Z(S)$ is not in Q and that $Z(S)$ is connected by Lemma 4.4. Now, since $S = QS_o$ and S/Q has dimension 1, S/Q has only two connected subgroups 1 and S/Q. Thus, $S = Z(S)Q$.
Since any two conjugates of S_o genetate M, any two conjugates of S generate G.
Hence we have $G = \langle S^G \rangle$ and so $G = \langle (Z(S))^G \rangle Q$ and $[(Z(S))^G, Q] = 1$.
Since $Z(S) \leq C_G(Q)$ and $C_G(Q)$ is normal in G, we have $\langle (Z(S))^G \rangle \leq C_G(Q)$.

Now, we have $G/C_G(Q) \cong Q/(Q \cap C_G(Q)) = Q/Z(Q)$ is nilpotent.
Thus $M \leq C_G(Q)$ by Lemma 4.2.
Therefore $[M, Q] = 1$.

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