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ON $M/M/3/3$ RETRIAL QUEUEING SYSTEM

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Abstract

We find a method finding the steady-state probabilities of $M/M/3/3$ retrial queueing system.

1. Introduction

Retrial queueing system are characterized by phenomenon that a customer who finds all severs busy upon arrival is obliged to leave the system and to re-initiate his request after a random period of time. Retrial queues have been widely used to model many problems in telephone switching system, telecommunication networks and computer networks.

The $M/M/c/c$ retrial queue has been extensively studied by Cohn [2]. He presents a model which includes the possibility that a customer who becomes discouraged may give up and leave the system. Analytic results are given for special cases only, including the case of one problem has been proposed by Riordan [10], but to estimate the accuracy of his method it is necessary to have exact formulas and conclusions. Keilson et. al. [8] established a recursive scheme for computing the state probabilities in the two servers case. Their algorithm is in base on a flow method similar to that employed in electrical engineering for the study of current flow in networks of resistor. Some explicit formulas for the steady-state probabilities have been derived by Jonin and Sedol [7]. The solution obtained provides immediate information on main characteristics of the two servers system, but no solution was found to simplifying the analytic expressions. The partial generating functions of the steady-state probabilities were expressed in terms of generalized hypergeometric functions by Hanschke [6]. In this paper it is shown that the steady-state probabilities of three servers

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system can be obtained by several recursive formulas. The results given below are supplementary to the investigation of Falin [2, 3, 4].

2. Model

We consider the retrial queueing system with three servers at which customers arrive according to a Poisson stream with rate λ. An arriving customer starts to get service immediately if upon his arrival there are free servers, and if upon his arrival all servers are busy he leaves the system and join the retrial group immediately in order to seek service facility free. The retrial time of each customer in the retrial group are assumed to be independent and exponentially distributed with density $f(x) = \nu \exp(-\nu x)$, $x \geq 0$. A customer who finds an idle server is served immediately. The service times of customers are assumed to be mutually independent and exponentially distributed with mean $\frac{1}{\mu}$.

3. Analysis of the steady-state probabilities

The states of the system can be described by a pair $X(t) = (L(t), Q(t))$, where $L(t)$ denotes the number of busy servers and $Q(t)$ the number of customers in the retrial group at time t. Then the stochastic process $\{X(t) : t \geq 0\}$ is a time-homogeneous Markov process with state space $\{0, 1, 2, 3\} \times \{0, 1, 2, \cdots\}$. And denote by (L, Q) the limiting random variable of $(L(t), Q(t))$ i.e. (L, Q) is the random variable of steady-state of $(L(t), Q(t))$. Define for $i = 0, 1, 2, 3$, $n = 0, 1, 2, \cdots$

$$\pi_{i,n} = P(L = i, Q = n).$$

The balance equations are given as follows; for $n = 0, 1, 2, \cdots$

$$(1) \qquad (\lambda + n\nu)\pi_{0,n} = \nu\pi_{1,n}$$

$$(2) \qquad (\lambda + \mu + n\nu)\pi_{1,n} = \lambda\pi_{0,n} + 2\mu\pi_{2,n} + (n+1)\nu\pi_{0,n+1}$$

$$(3) \qquad (\lambda + 2\mu + n\nu)\pi_{2,n} = \lambda\pi_{1,n} + 3\mu\pi_{3,n} + (n+1)\nu\pi_{1,n+1}$$

$$(4) \qquad (\lambda + 3\mu)\pi_{3,n} = \lambda\pi_{2,n} + (n+1)\nu\pi_{2,n+1} + \lambda\pi_{3,n-1}$$

where $\pi_{3,-1} = 0$.

To study the equations (1)–(4), we make use of a partial generating function technique. For $i = 0, 1, 2$ and 3, define the partial generating functions

$$(5) \qquad G_i(z) = \sum_{n=0}^{\infty} \pi_{i,n} z^n$$

Since $\sum_{i,n} \pi_{i,n} = 1$ these functions converge for at least all $|z| \leq 1$. When we multiply both sides of (1)–(4) by z^n and sum over all n, we obtain

$$\lambda G_0(z) + \nu z G_0'(z) = \mu G_1(z) \tag{6}$$

$$\lambda G_1(z) + \mu G_1(z) + \nu z G_1'(z) = \lambda G_0(z) + 2\mu G_2(z) + \nu G_0'(z) \tag{7}$$

$$\lambda G_2(z) + 2\mu G_2(z) + \nu z G_2'(z) = \lambda G_1(z) + 3\mu G_3(z) + \nu G_1'(z) \tag{8}$$

$$\lambda G_3(z) + 3\mu G_3(z) = \lambda G_2(z) + \nu G_2'(z) + \lambda z G_3(z) \tag{9}$$

where the prime indicates the derivatives with respect to z. Adding all equations and dividing the resulting relation by $(1-z)$ we get

$$\nu\{G_0'(z) + G_1'(z) + G_2'(z)\} = \lambda G_3(z). \tag{10}$$

By repeated substitutions and differentiations one deduces from (6)–(9) that

$$\mu G_1'(z) = (\lambda + \nu) G_0'(z) + \nu z G_0''(z) \tag{11}$$

$$2\mu^2 G_2(z) = \lambda^2 G_0(z) + \{(2\lambda + \mu + \nu)z - \mu\}\nu G_0'(z) + \nu^2 z^2 G_0''(z) \tag{12}$$

$$\begin{aligned} 2\mu^2 G_2'(z) &= (\lambda^2 + 2\lambda\mu + \mu\nu + \nu^2) G_0'(z) \\ &\quad + \{(2\lambda + \mu + 3\nu)z - \mu\}\nu G_0''(z) + \nu^2 z^2 G_0''(z) \end{aligned} \tag{13}$$

$$\begin{aligned} 6\mu^3 G_3(z) &= \lambda^3 G_0(z) + \{(3\lambda^2 + 3\lambda\mu + 3\lambda\nu + 2\mu^2 + 3\mu\nu + \nu^2)z \\ &\quad - 3\lambda\mu - 2\mu^2 - 2\mu\nu\}\nu G_0'(z) \\ &\quad + \{3(\lambda + \mu + \nu)z^2 - 3\mu z\}\nu^2 G_0''(z) + \nu^3 z^3 G_0'''(z) \end{aligned} \tag{14}$$

Substituting (6), (11)-(14) into (10) leads to a third order differential equation for $G_0(z)$;

$$\begin{aligned} &\nu^3(\lambda z - 3\mu) z^2 G_0'''(z) \\ &+ \nu^2\{3\lambda(\lambda + \mu + \nu)z^2 - 9\mu(\lambda + \mu + \nu)z - 3\mu^2\} G_0''(z) \\ &+ \nu\{\lambda(3\lambda^2 + 3\lambda\mu + 3\lambda\nu + 2\mu^2 + 3\mu\nu + \nu^2)z \\ &\quad - (6\lambda^2\mu + 8\lambda\mu^2 + 8\lambda\mu\nu + 9\mu^2\nu + 6\mu^3 + 3\mu\nu^2)\} G_0'(z) \\ &+ \lambda^4 G_0(z) = 0. \end{aligned} \tag{15}$$

Let

$$\begin{aligned}
&A_0 = \lambda\nu^3, \qquad A_1 = 3\mu\nu^3, \qquad A_2 = 3\lambda(\lambda+\nu+\mu)\nu^2,\\
&A_3 = 9\mu(\lambda+\nu+\mu)\nu^2, \qquad A_4 = 3\mu^2\nu^2,\\
&A_5 = \lambda(3\lambda^2+3\lambda\mu+3\lambda\nu+2\mu^2+3\mu\nu+\nu^2)\nu,\\
&A_6 = (6\lambda^2\mu+8\lambda\mu^2+8\lambda\mu\nu+9\mu^2\nu+6\mu^3+3\mu\nu^2),\\
&A_7 = \lambda^4.
\end{aligned}$$

Then we can rewrite (15) as follows;

$$\begin{aligned}
&(A_0 z^3 - A_1 z^2)G_0'''(z) + (A_2 z^2 - A_3 z - A_4)G_0''(z)\\
&+ (A_5 z - A_6)G_0'(z) + A_7 G_0(z) = 0.
\end{aligned}$$

By substituting $G_0(z) = \sum_{n=0}^{\infty} \pi_{0,n} z^n$, we get the equation

$$\begin{aligned}
&A_0 \sum_{n=3}^{\infty} n(n-1)(n-2)\pi_{0,n} z^n - A_1 \sum_{n=2}^{\infty}(n+1)n(n-1)\pi_{0,n+1} z^n\\
&+ A_2 \sum_{n=2}^{\infty} n(n-1)\pi_{0,n} z^n - A_3 \sum_{n=1}^{\infty}(n+1)n\pi_{0,n+1} z^n\\
&- A_4 \sum_{n=2}^{\infty}(n+2)(n+1)\pi_{0,n+2} z^n + A_5 \sum_{n=1}^{\infty} n\pi_{0,n} z^n\\
&- A_6 \sum_{n=0}^{\infty}(n+1)\pi_{0,n+1} z^n + A_7 \sum_{n=2}^{\infty} \pi_{0,n} z^n = 0.
\end{aligned} \tag{16}$$

By equating the coefficient of z^n in (16) being zero, we obtain the recursive relation for $\{\pi_{0,n}\}$ for $n = 0, 1, 2, \cdots$

$$\begin{aligned}
&\{A_0 n(n+1)(n-2) + A_2 n(n-1) + A_5 n\}\pi_{0,n}\\
&- \{A_1 n(n+1)(n-1) + A_3 n(n+1) + A_6(n+1)\}\pi_{0,n+1}\\
&- A_4(n+2)(n+1)\pi_{0,n+2} = 0.
\end{aligned} \tag{17}$$

For simplicity, let

$$\begin{aligned}
X_{n+1} &= \frac{A_1 n(n-1) + A_3 n + A_6}{A_4(n+2)},\\
Y_n &= \frac{A_0 n(n-1)(n-2) + A_2 n(n-1) + A_5 n}{A_4(n+2)(n+1)}.
\end{aligned}$$

Then

$$\pi_{0,n+2} = -X_{n+1}\pi_{0,n+1} + Y_n\pi_{0,n}. \tag{18}$$

By succesive substitution in (18), $\pi_{0,n}$ can be expressed by linear combination of $\pi_{0,0}$ and $\pi_{0,1}$ as follows

$$\pi_{0,n} = -H_n\pi_{0,1} + K_n\pi_{0,0}. \tag{19}$$

H_{n+2} and K_{n+2} have the following recursive formulas

$$H_{n+2} = -X_{n+1}H_{n+1} + Y_nH_n, \tag{20}$$

$$K_{n+2} = -X_{n+1}K_{n+1} + Y_nK_n. \tag{21}$$

In this expression, we have the below proposition:

PROPOSITION 1. *The* $\lim_{n\to\infty}\frac{K_n}{H_n}$ *exists.*

Proof. Since $\pi_{0,n}$ is the probability of the steady-state,

$$\begin{aligned} 0 &= \lim_{n\to\infty}\pi_{0,n} \\ &= \lim_{n\to\infty}(-H_n\pi_{0,1} + K_n\pi_{0,0}) \\ &= \lim_{n\to\infty}H_n(-\pi_{0,1} + \frac{K_n}{H_n}\pi_{0,0}) \end{aligned}$$

We know $\lim_{n\to\infty}H_n = \infty$ since $\lim_{n\to\infty}X_n = \infty$, $\lim_{n\to\infty}Y_n = \infty$ and X_n, Y_n are different type functions of n. So $\lim_{n\to\infty}(-\pi_{0,1} + \frac{K_n}{H_n}\pi_{0,0}) = 0$. Since both $\pi_{0,0}$ and $\pi_{0,1}$ are not zero, $\lim_{n\to\infty}\frac{K_n}{H_n}$ must be exist. □

Let $L = \lim_{n\to\infty}\frac{K_n}{H_n}$. Then by Proposition 1, $\pi_{0,1} = L\pi_{0,0}$. Let $W_n = -H_nL + K_n$. Then

$$\begin{aligned} \pi_{0,n} &= -H_n\pi_{0,1} + K_n\pi_{0,0} \\ &= (-H_nL + K_n)\pi_{0,0} \\ &= W_n\pi_{0,0} \end{aligned} \tag{22}$$

By (1)

$$\pi_{1,n} = \frac{\lambda + n\nu}{\mu}\pi_{0,n} = \frac{\lambda + n\nu}{\mu}W_n\pi_{0,0}.$$

By (2)

$$\pi_{2,n} = [\{(\lambda+\mu+n\nu)\frac{\lambda+n\nu}{2\mu^2} - \frac{\nu}{2\mu}\}W_n - \frac{(n+1)\nu}{2\mu}W_{n+1}]\pi_{0,0}$$

By (3)

$$\pi_{3,n} = [\{\frac{(\lambda+2\mu+n\nu)\{(\lambda+\mu+n\nu)(\lambda+n\nu)-\lambda\mu\}}{6\mu^3} - \frac{\lambda(\lambda+n\nu)}{3\mu^2}\}W_n - \frac{\{(\lambda+2\mu+n\nu)-2(\lambda+n\nu)\}(n+1)\nu}{6\mu^2}W_{n+1}]\pi_{0,0}$$

$$\pi_{3,n} = [\frac{(\lambda+2\mu+n\nu)(\lambda^2+2n\lambda\mu+n\mu\lambda+n^2\nu^2)-2\lambda\mu(\lambda+n\nu)}{6\mu^3}W_n - \frac{(n+1)(2\mu-\lambda-n\nu)}{6\mu^2}W_{n+1}]\pi_{0,0}.$$

By the normalizing condition $\sum_{i,n}\pi_{i,n}=1$ we obtain

$$\pi_{0,0} = [\sum_{n=0}^{\infty}[\frac{1}{6\mu^3}\{n^3\nu^3+3n^2\nu^2(\lambda+2\mu)+n\nu(3\lambda^2+11\mu^2+9\lambda\mu) + \lambda^3+\lambda^2\mu+9\lambda\mu^2+6\mu^3\}W_n - \frac{1}{6\mu^2}(n+1)\nu(5\mu-\lambda-n\nu)W_{n+1}]]^{-1}.$$

Thus we have the following theorem;

THEOREM 2. *The joint probabilities of (L, Q) are given by*

$$\pi_{0,n} = W_n\pi_{0,0},$$

$$\pi_{1,n} = \frac{\lambda+n\nu}{\mu}W_n\pi_{0,0},$$

$$\pi_{2,n} = [\{(\lambda+\mu+n\nu)\frac{\lambda+n\nu}{2\mu^2} - \frac{\nu}{2\mu}\}W_n - \frac{(n+1)\nu}{2\mu}W_{n+1}]\pi_{0,0},$$

$$\pi_{3,n} = [\frac{(\lambda+2\mu+n\nu)(\lambda^2+2n\lambda\nu+n\mu\nu+n^2\nu^2)-2\lambda\mu(\lambda+n\nu)}{6\mu^3}W_n - \frac{(n+1)\nu(2\mu-\lambda-n\nu)}{6\mu^2}W_{n+1}]\pi_{0,0}.$$

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