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FUZZY MAXIMAL P–IDEALS OF BCI–ALGEBRAS

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Abstract

Our task will be to set up a fuzzy maximal p-ideal in BCI-algebras. We construct a new fuzzy p-ideal from old. We also prove that every fuzzy maximal p-ideal is normalized, and takes only the values $\{0, 1\}$.

The concept of a fuzzy set, which was introduced by Zadeh [11], is applied to BCK-algebras by Xi [10]. In [12] Zhang and Hao introduced the concept of p-ideals of BCI-algebras and obtained a number of its properties. The fuzzification of p-ideals is stated from the first author with Meng [7]. Also the first author, in [5], considered normalized fuzzy p-ideals and fuzzy characteristic p-ideals.

In this paper, we discuss a fuzzy maximal p-ideals in BCI-algebras. We verify that every fuzzy maximal p-ideal is normalized, and takes only the values $\{0, 1\}$. We also construct a new fuzzy p-ideal from old.

A *BCI-algebra* is an algebra $(X; *, 0)$ of type (2, 0) satisfying the following axioms:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply that $x = y$,

for all $x, y, z \in X$. A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x * y = 0$.

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A fuzzy set in a set X is a function $A : X \to [0,1]$. Let A and B be fuzzy sets in a set X. Then we define

$$A = B \iff A(x) = B(x) \text{ for all } x \in X.$$

$$A \subseteq B \iff A(x) \leq B(x) \text{ for all } x \in X.$$

In the rest of the present paper, X shall denote a BCI-algebra, and $\mathrm{Im}(A)$ denote the image set of a fuzzy set A in X.

DEFINITION 1 ([12]). A non-empty subset I of X is called a *p-ideal* of X if

(i) $0 \in I$,
(ii) $(x * z) * (y * z) \in I$ and $y \in I$ imply that $x \in I$ for all $x, y, z \in X$.

We shall write $a \wedge b$ for $\min\{a, b\}$ where a and b are two real numbers.

DEFINITION 2 ([7]). A fuzzy set A in X is called a *fuzzy p-ideal* of X if

(i) $A(0) \geq A(x)$ for any $x \in X$,
(ii) $A(x) \geq A((x * z) * (y * z)) \wedge A(y)$ for any $x, y, z \in X$.

LEMMA 3 ([7]). *If a fuzzy set A in X is a fuzzy p-ideal of X, then the set $X_A = \{x \in X | A(x) = A(0)\}$ is a p-ideal of X.*

Let Λ be a nonempty subset of $[0,1]$.

THEOREM 4. *Let $\{I_\lambda | \lambda \in \Lambda\}$ be a collection of p-ideals of X such that $X = \bigcup_{\lambda \in \Lambda} I_\lambda$ and for all $\alpha, \beta \in \Lambda$, $\alpha > \beta$ if and only if $I_\alpha \subset I_\beta$. Define a fuzzy set A in X by $A(x) = \sup\{\lambda \in \Lambda | x \in I_\lambda\}$ for all $x \in X$. Then A is a fuzzy p-ideal of X.*

Proof. Since $0 \in I_\lambda$ for all $\lambda \in \Lambda$, clearly $A(0) \geq A(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $A((x * z) * (y * z)) = m$ and $A(y) = n$. Without loss of generality we may assume that $m \leq n$. To Prove $A(x) \geq A((x * z) * (y * z)) \wedge A(y)$, we consider the following three cases:

$$\text{(i) } \lambda \leq m, \text{ (ii) } m < \lambda \leq n \text{ and (iii) } \lambda > n.$$

Case (i) implies that $(x * z) * (y * z) \in I_\lambda$ and $y \in I_\lambda$. Since I_λ is a p-ideal of X, it follows that $x \in I_\lambda$, so that

$$A(x) = \sup\{\lambda \in \Lambda | x \in I_\lambda\} \geq m = A((x * z) * (y * z)) \wedge A(y).$$

For the case (ii), we have $(x * z) * (y * z) \notin I_\lambda$ and $y \in I_\lambda$. It follows that either $x \in I_\lambda$ or $x \notin I_\lambda$. If $x \in I_\lambda$, then

$$A(x) = n \geq A((x * z) * (y * z)) \wedge A(y).$$

If $x \notin I_\lambda$ then $x \in I_\delta - I_\lambda$ for some $\delta < \lambda$, and so

$$A(x) > m = A((x * z) * (y * z)) \wedge A(y).$$

Finally case (iii) implies $(x * z) * (y * z) \notin I_\lambda$ and $y \notin I_\lambda$. Then we also have either $x \in I_\lambda$ or $x \notin I_\lambda$. If $x \in I_\lambda$, then obviously

$$A(x) \geq A((x * z) * (y * z)) \wedge A(y).$$

If $x \notin I_\lambda$ then $x \in I_\epsilon - I_\lambda$ for some $\epsilon < \lambda$, and thus

$$A(x) \geq m = A((x * z) * (y * z)) \wedge A(y).$$

This completes the proof.

DEFINITION 5 ([5]). A fuzzy p-ideal A of X is said to be normalized if $A(0) = 1$.

LEMMA 6 ([5]). *Let I be a p-ideal of X. Define a mapping $A_I : X \to [0,1]$ by*

$$A_I(x) = \begin{cases} 1 & \textit{if } x \in I, \\ 0 & \textit{if } x \notin I. \end{cases}$$

Then A_I is a normalized fuzzy p-ideal of X and $X_{A_I} = I$.

THEOREM 7. *Let I and J be p-ideals of X. Then $I \subseteq J$ if and only if $A_I \subseteq A_J$.*

Proof. Let I and J be p-ideals of X such that $I \subseteq J$, and let $x \in X$. If $x \in I$, then $x \in J$ and so $A_I(x) = 1 = A_J(x)$. If $x \in J - I$, then $A_I(x) = 0 < 1 = A_J(x)$. If $x \in X - J$, then $A_I(x) = 0 = A_J(x)$. Hence $A_I \subseteq A_J$.

Conversely assume that $A_I \subseteq A_J$ for any p-ideals I and J of X. Let $x \in I$. Then $A_J(x) \geq A_I(x) = 1$, and hence $A_J(x) = 1$. This means that $x \in J$. The proof is complete.

The following is easily verified.

PROPOSITION 8. *If A and B are normalized fuzzy p-ideals of X, then so is $A \cap B$.*

LEMMA 9 ([5]). *Let A be a fuzzy p-ideal of X and A^+ a fuzzy set in X defined by*

$$A^+(x) = A(x) + 1 - A(0)$$

for all $x \in X$. Then A^+ is a normalized fuzzy p-ideal of X and $A \subseteq A^+$.

LEMMA 10 ([5]). *If A is a fuzzy p-ideal of X satisfying $A^+(x) = 0$ for some $x \in X$, then $A(x) = 0$ also.*

LEMMA 11 ([5]). *A fuzzy p-ideal of X is normalized if and only if $A^+ = A$.*

Using a given fuzzy p-ideal, we will construct a new fuzzy p-ideal. Let $\alpha \geq 0$ be a real number. If $m \in [0,1]$, m^α shall mean the positive root in case $\alpha < 1$. We define $A^\alpha : X \to [0,1]$ by $A^\alpha(x) = (A(x))^\alpha$ for all $x \in X$.

THEOREM 12. *If A is a fuzzy p-ideal of X, then A^α is also a fuzzy p-ideal of X and $X_{A^\alpha} = X_A$.*

Proof. Note that $A^\alpha(0) = (A(0))^\alpha \geq (A(x))^\alpha = A^\alpha(x)$ for all $x \in X$. Let $x, y, z \in X$. Then

$$\begin{aligned} A^\alpha(x) &= (A(x))^\alpha \\ &\geq (A((x*z)*(y*z)) \wedge A(y))^\alpha \\ &= (A((x*z)*(y*z)))^\alpha \wedge (A(y))^\alpha \\ &= A^\alpha((x*z)*(y*z)) \wedge A^\alpha(y). \end{aligned}$$

Hence A^α is a fuzzy p-ideal of X. Now

$$\begin{aligned} X_{A^\alpha} &= \{x \in X | A^\alpha(x) = A^\alpha(0)\} \\ &= \{x \in X | (A(x))^\alpha = (A(0))^\alpha\} \\ &= \{x \in X | A(x) = A(0)\} \\ &= X_A. \end{aligned}$$

The following corollary is obvious.

COROLLARY 13. *If A is a normalized fuzzy p-ideal of X then A^α is also a normalized fuzzy p-ideal of X.*

Let $\mathcal{I}_p(X)$ (resp. $\mathcal{NF}_p(X)$) denote the set of p-ideals (resp. normalized fuzzy p-ideals) of X. We can define functions $\phi : \mathcal{I}_p(X) \to \mathcal{NF}_p(X)$ and $\psi : \mathcal{NF}_p(X) \to \mathcal{I}_p(X)$ by $\phi(I) = A_I$ and $\psi(A) = X_A$. Then $\psi\phi = 1_{\mathcal{I}_p(X)}$ and $\phi\psi(A) = \phi(X_A) = A_{X_A} \subseteq A$.

THEOREM 14. *If I and J are p-ideals of X, then $A_{I\cap J} = A_I \cap A_J$. If A and B are normalized fuzzy p-ideals of X, then $X_{A\cap B} = X_A \cap X_B$. Thus $\phi(I\cap J) = \phi(I)\cap\phi(J)$ and $\psi(A\cap B) = \psi(A)\cap\psi(B)$.*

Proof. Let $x \in X$. If $x \in I\cap J$, then $A_{I\cap J}(x) = 1$. From $x \in I$ and $x \in J$ it follows that $A_I(x) = 1 = A_J(x)$. Hence

$$A_{I\cap J}(x) = 1 = A_I(x) \wedge A_J(x) = (A_I \cap A_J)(x).$$

If $x \notin I\cap J$ then $x \notin I$ or $x \notin J$. Thus

$$A_{I\cap J}(x) = 0 = A_I(x) \wedge A_J(x) = (A_I \cap A_J)(x).$$

Therefore $A_{I\cap J} = A_I \cap A_J$. Let $A, B \in \mathcal{NF}_p(X)$. Then

$$\begin{aligned} X_{A\cap B} &= \{x \in X | (A\cap B)(x) = (A\cap B)(0)\} \\ &= \{x \in X | A(x) \wedge B(x) = 1\} \\ &= \{x \in X | A(x) = 1 \text{ and } B(x) = 1\} \\ &= \{x \in X | A(x) = 1\} \cap \{x \in X | B(x) = 1\} \\ &= \{x \in X | A(x) = A(0)\} \cap \{x \in X | B(x) = B(0)\} \\ &= X_A \cap X_B. \end{aligned}$$

Thus

$$\phi(I\cap J) = A_{I\cap J} = A_I \cap A_J = \phi(I)\cap\phi(J)$$

and

$$\psi(A\cap B) = X_{A\cap B} = X_A \cap X_B = \psi(A)\cap\psi(B).$$

This completes the proof.

DEFINITION 15. A fuzzy p-ideal A of X is called a *fuzzy maximal p-ideal* if

(i) A is non-constant,

(ii) A^+ is a maximal element of the poset $(\mathcal{NF}_p(X), \subseteq)$.

LEMMA 16 ([5]). *Let A be a non-constant normalized fuzzy p-ideal of X such that it is a maximal element of $(\mathcal{NF}_p(X), \subseteq)$. Then* Im$(A)$= $\{0, 1\}$.

THEOREM 17. *If A is a fuzzy maximal p-ideal of X, then*

(i) *A is normalized.*

(ii) $\mathrm{Im}(A)=\{0,1\}$.

(iii) $A_{X_A}=A$.

(iv) *X_A is a maximal p-ideal of X.*

Proof. Let A be a fuzzy maximal p-ideal of X. Then A^+ is a non-constant maximal element of $\mathcal{NF}_p(X)$. It follows from Lemma 16 that $\mathrm{Im}(A^+)=\{0,1\}$. Note that $A^+(x)=1$ if and only if $A(x)=A(0)$, and $A^+(x)=0$ if and only if $A(x)=A(0)-1$. By Lemma 10 we have $A(x)=0$, and so $A(0)=1$. Hence A is normalized, and by Lemma 11 we get $A^+=A$. Therefore $\mathrm{Im}(A)=\mathrm{Im}(A^+)=\{0,1\}$, proving (i) and (ii).

(iii) Clearly $A_{X_A}\subseteq A$ and $\mathrm{Im}(A_{X_A})=\{0,1\}$. Let $x\in X$. If $A(x)=0$, then obviously $A\subseteq A_{X_A}$. If $A(x)=1$, then $x\in X_A$ and so $A_{X_A}(x)=1$. This proves that $A\subseteq A_{X_A}$.

(iv) Noticing that A is non-constant, we know that X_A is a proper p-ideal of X. Let I be a p-ideal of X such that $X_A\subseteq I$. Then $A=A_{X_A}\subseteq A_I$ by Theorem 7 and (iii) above. Since $A, A_I\in\mathcal{NF}_p(X)$ and since $A=A^+$ is a maximal element of $\mathcal{NF}_p(X)$, we have that either $A=A_I$ or $A_I=1$ where $1: X\to[0,1]$ is a fuzzy set defined by $1(x)=1$ for all $x\in X$. The latter case implies that $I=X$. If $A=A_I$, then $X_A=X_{A_I}=I$ by Lemma 6. Hence either $I=X$ or $X_A=I$, proving that X_A is a maximal p-ideal of X.

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