# Volatility Puzzle, Equity Premium Puzzle, And Mean Reversion; Are They Interrelated Phenomena?

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## ABSTRACT

According to recent empirical studies, there is a systematic pattern in temporal behaviors of asset returns , and that systematic pattern is related to the business cycle. I propose a model which captures this evidence. This is done by considering a state dependent preference structure where state dependency is related to the business cycle. In this setting, the three main puzzles (i.e., the volatility puzzle, the equity premium puzzle, mean reversion) are understood as interrelated behaviors.

## I. INTRODUCTION

In the standard utility-based asset pricing model, a rational agent maximizes expected utility where preferences are assumed to be time-separable and state-independent. one of the concerns with this model is its lack of volatility. To see this point more clearly, suppose no exogenous stochastic shocks are present.

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Then, the system tends to a fixed point, implying inherent stability. Even if exogenous stochastic shocks are introduced into the system, the resulting dynamics are more stable than intuition and empirical evidence would suggestl.[1] Thus, an improved asset pricing model needs an ingredient which can counteract the smoothing mechanism[2] inherent in the standard model.

There are several approaches to constructing models which can overcome this deficiency. One approach Delong, Bradford, Schleifer, and Summers (1988) -- is to eliminate 'rationality.' If the agent is not rational by being short-sighted, more variability would result. Another approach, by Epstein and Zin (1989), has the agent more concerned about the timing of the resolution. of uncertainty. This frustrates the intertemporal arbitrage operation, and thus distorts the smoothing mechanisin. Yet another modelling approach, used by Constantinides (1990), introduces the idea of 'habit formation' on the time domain. A positive feedback loop arises from this formulation which uses a modified preference structure. In all of these different attempts, the common theme is to disable the smoothing mechanism inherent in the dynamic structure of the standard model.

Dropping the rationality assumption -- which is a time honored and generally useful assumption in economic models -- seems like a severe stop. An alternative explanation, which retains rationality, to account for the greater variability found in the empirical studies would be more desirable. Recently, there have been a number of empirical studies which suggest an alternative explanation. For instance, Ferson and Merrick (1987) found that the failure of the standard model is associated with the business cycle. By controlling this cyclical effect, they found less evidence against the standard model in non-recessionary periods but more evidence against it in recessionary periods. Schwert (1988) related stock

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11) There are many empirical studies which tend to support this observation. See, for example, Shiller (1981), Hansen and Singleton (1982, 1983), and Grossman, Melino, and Shiller (1985).

2) Several examples of the smoothing mechanism might be: (1) when price goes up, demand goes down and supply goes up, which in turn leads to a fall in price; (2) when the price of consumption is abnormally high, people postpone consumption, which drives down the price of consumption, (3) when the current prices of inputs for production are viewed as abnormally high # factories tend to postpone production, thereby lowering the input price.

market volatility to some macro-economic variables, found correlations, and also found that both macro-economic variables and the stock market are more volatile in recessionary periods.[3] Others have found that mean reversion in asset returns is related to stock market volatility,[4] and the effect seems more significant in recessionary periods (Schwert, 1988; Kim, Nelson, and Startz, 1988).

Overall there seems to be a cyclical pattern, related to the business cycle, in observed temporal behavior of asset returns. Thus one could argue that the temporal behavior of asset returns is state dependent, where the state dependency arises from and is related to the business cycle. But, the temporal behavior of asset returns should be explained within an asset pricing model. This suggests that the inclusion of some state variable which is correlated with the business cycle should be a part of an alternative asset pricing model.

We propose a simple model which captures this idea. In the model, the investor's preference structure, which includes a state variable, is correlated with the business cycle. Since the proposed preference is state dependent, the resulting dynamics will be more volatile, and thus this implies blunting the smoothing mechanism inherent in the standard model. In addition to adding insight with regard to the "volatility puzzle," the correlation structure in the model becomes a driving force that resolves both the "equity premium puzzle" and explains the mean reversion of asset prices that have been observed in the literature . An interesting finding is that all of these puzzles are driven by the same correlation structure, which means that these are, in fact, interrelated phenomena.

3) Instead of applying a formal test, be informal, and yet have a more close look at the data.

4) Campbell and shiller (1988), for instance, claim that the mean reversion and volatility puzzle are interrelated phenomena.

## II. THE MODEL

We assume an economy with a single agent, a single consumption good, and two financial assets (a risky stock and a risk-free bond). since we deal with a single agent, the equilibrium condition for the model is zero net borrowing (or lending). We don't model technology directly. If investments are fluid, we will reach equilibrium through shifts in the investment mix. If investments have fixed state-dependent payoffs, we will reach equilibrium through shifts in asset prices. The first-order conditions should hold for any combination of fluid and fixed investments. With a single agent, wealth is just a share of all the assets in the economy. So we can speak of an agent as choosing a mix of assets with borrowing or lending. All variables measuring quantities are per cpaital.

The problem of choosing optimal portfolio and consumption rules in the proposed model is formulated as the solution to

$$\underset{C,\,x}{Max} \quad [E_t \int_t^{\infty} e^{-\rho(\tau-t)} U(C,N)\, d\tau] \qquad (1)$$

Subject to

$$dW=(rW+ax-C)\,dt+sx\,dz_1 \qquad (2),$$
$$dN=f\,dt+g\,dz_N \qquad (3),$$

where $E[\, dz_1 dz_N ] = \phi dt$. Here U(C, N) is the instantaneous direct utility function, C is consumption, N is a state variable which follows a wiener process, with mean f and the standard deviation g, where f and g are positive constants. W is wealth, $\rho$ is the time preference rate, $\omega$ is a fraction invested in the risky asset, $x=\omega W$ is the amount of wealth invested in the risky asset, and r is the rate of return on risk-free asset. The rate return on the risky asset follows the Ito process $dP/P = \alpha dt + s dz_1$, with expected return $\alpha$, volatility $s^2$, and $dz_1$ a Wiener process. $a = \alpha - r$ is the equity (or risk) premium.

The business cycle, in this simple economy, is modelled in terms of the stochastic process $dz_1$,[5] the between $dz_1$ and $dz_N$ is $\phi dt$+dt.

Applying the Hamilton/Jacobi/Bellman equation and Ito's Lemma, the indirect utility function V(W, N) will satisfy

$$\rho V(W,N) = Max\,[U(C,N) + V_W(rW + ax - C) + (\frac{1}{2})V_{WW}s^2x^2 + V_N f + (\frac{1}{2})V_{NN}g^2 + V_{WN}sxg\Phi] \tag{4}$$

where subscripts refer to derivatives.
The first-order conditions are

$$0 = -V_W(W,N) + U_C(C,N) \tag{5}$$
$$0 = aV_W + s^2xV_{WW} + sg\Phi V_{WN} \tag{6}$$

In equilibrium, an investor holds all of his wealth in assets, and net borrowing will be zero. Thus, the equilibrium condition is

$$X(W,N) = W \tag{7}$$

5)The utility-based asset pricing model is in fact a decentralized form of Neoclassical-based growth model with a stock market in it. The neoclassical-based business cycle, on the other hand, is just a manifestation of stochastic growth in a Neoclassical-based growth model. Here, the shocks to the supply side are stressed as central driving forces in investigating the dynamic elements of the transformation mechanism in which the shocks to the supply side propagate. According to this Neoclassical-based model, the shocks to the supply side becomes transformed into something which in turn follows the business cycle. In this model, therefore, $dz_1$ can be thought of as the shock to the supply side, which is in turn transformed into W (market portfolio) that follows the business cycle. Hence, market portfolio cum business cycle.

In order to get a closed form solution, we need another refinement. Let's assume that

$$U(C,N)=e^{-\lambda N}\frac{C^{1-\delta}}{1-\delta} \tag{8}$$

where $\delta$ is Relative Risk Aversion (RRA) and $\lambda$ is a constant. This follows first a value function of the following form:

$$V(W,N)=e^{-\theta N}\frac{W^{1-\delta}}{1-\delta} \tag{9}$$

With equation (8), equation (5) becomes

$$C^{*}=e^{([\frac{\theta-\lambda}{\delta}]N)}W \tag{10}$$,

*from which it follows that,*

$$Var(\frac{dC}{C})=s^{2}+([\frac{\theta-\lambda}{\delta}]g)^{2}+2(\frac{\theta-\lambda}{\delta})\phi sg \tag{11}$$

where $s^2$ = Var(dW/W), and $\phi sg$ = Cov(dW/W, dN). In equation (11), the first two terms of the consumption volatility are always positive. For the volatility of consumption to be less than that of wealth, the last covariance term has to be negative and must exceed the second term in absolute value. This requires, $\phi$, the correlation structure between dN and dW, to be negative, assuming $\theta > \lambda$. In other words, the negative correlation structure is a necessary condition for solving the volatility puzzle. In addition, if $-2s\phi > [(\theta-\lambda)/\delta]\, g$ and $\theta > \lambda$, the volatility of consumption is less than that of wealth.

Given eqqation (8), the portfolio choice equation (6) with equilibrium condition (7) becomes

$$\omega^{*} = 1 = \frac{a}{\delta s^{2}} + \frac{\theta}{\delta s^{2}} \phi sg \qquad (12)$$

where $\phi sg = Cov(dW/W, N)$. In order for a high equity premium to be consistent with the low RRA, tha covariance term on the right-hand-side of equation (12) has to be negative. This requires a correlation between N and W be negative, assuming $\theta > 0$. A solution to the equity premium puzzle, therefore, must satisfy $\phi < 0$, where $\theta > 0$.

With equation (8), it follows that the derivative of the equity premium, "a", with respect to the state variable, N, is[6]

$$a_{N} = 2\left(\frac{\delta}{\delta-1}\right)\left(\frac{\theta-\lambda}{\delta}\right) e^{\left(\left[\frac{\theta-\lambda}{\delta}\right] N\right)} \qquad (13)$$

Mean reversion can also be justified if RRA is greater than 1. Suppose a negative correlation between N and W. Then W goes up when N goes down. As N goes down, we can see from equation (13) that an goes down as well. In other words, a wealth increase (decrease) is accompanied by a decrease (increase) in risk premium,[7] which then implies mean reversions.[8] A solution to the mean

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6) see Appendix 1.

7) There are recent empirical studies (including Fama and French (1989), Kandel and stambaugh (1990), Ferson and Harvey (1990)) which show that the risk premium is related to business conditions, low when conditions are good, and high when conditions are bad.

8) Note here that it is assumed that $r_N$ and $s_N$ are equal to 0. This assumption requires justification. One explanation of rnjo is that the interest rate is easier to measure than the risk premium. Thus, investors will respond more to interest rate changes than to risk premium changes. Their responses, in turn p will dampen the interest rate changes more than the risk premium changes. As a result, we can assume $r_n = 0$. Concerning $s_N = 0$, there are some stylized facts that might suggest that sn is not equal to 0. For example, it is argued that s is known as conditionally heteroskedastic, and unconditionally leptokurtic. The discrete time (G) ARCH model

reversion, therefore, must satisfy $\phi<0$, where $\delta>1$ and $\theta>\lambda$.

Theorem 1: There exist parameter values which resolve the volatility puzzle, equity premium puzzle, a mean reversion, simultaneously.

Proof will be provided upon request.

Theorem 1 says that the negative correlation in fact solves all the puzzles simultaneously. The volatility puzzle, equity premium puzzle, and mean reversion have been found and described independently in empirical work. Nevertheless, they may be interrelated phenomena. In the proposed model, the interrelation comes because all the puzzles are solved by the same sign of correlation structure.

## III. IMPLICATIONS OF THE PROPOSED MODEL

According to recent empirical studies,[9] there is a cyclical pattern in the temporal behavior of asset returns where cyclicality is related to the business cycle. We therefore propose a model where people behave cyclically. This is captured by a state dependent preference structure where a state variable is correlated with the market portfolio cum business cycle. Then, a negative correlation structure becomes the key in explaining the three maj or puzzles. A negative correlation implies that people are more relax when the econoiny is in boom, but become more desperate when the economy is in recession[10]. When people have such a

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fits well in this case because it creates simulated data series that exhibit persistence and busting. But, these facts are associated with high frequency data (such as daily data). Mean reversion, on the other hand, is a long horizon phenomenon which tends to follow business cycle frequency. In investigating the long run behavior of asset prices, therefore, we can assume $s_N = 0$.

9) Refer to the studies by Ferson Merrick (1987), Schwert (1988), Fama and French (1988, 1989), Ferson and Harvey Kandel and Stambaugh (1990), among others.

10) With the concave utility function, the marginal utility is low in boom and hiqh in recession. But,

cyclical preference structure, the temporal behavior of asset returns becomes cyclical. Furthermore, since all aforementioned major puzzles are simultaneously explained by the same correlation structure imbedded within the proposed state dependent preference structure, we can argue that those puzzles are in fact an interrelated phenomena.

It is possible to put an intuitive description to these interrelated phenomena . consider investors at the trough of the business cycle expecting a rise. Since rises are highly persistent,[11] investor would expect a bull market to last for some time. One would also expect that the marginal utility of consumption to fall and the demand for securities to rise along with security prices. Contrast this with investors at the peak anticipating a fall in economic activity. One would expect that marginal utility of consumption would rise causing a shift away from securities and thus price declines. The point is that this cyclical change in the attitude should magnify swings in stock returns over the business cycle. The ultimate roots of swings in stock returns relate back to swings in economic activity in general. Hence mean reverslon.

We also have high volatility in the stock market, not because the market is crowded with noise traders, but because investors have cyclically volatile preferences. In the face of a linear budqet constraint and a concave preference structure,[12] people still want to smooth out their consumption stream. But with the proposed preference structure, they will be affected by a cyclically volatile state variable. Think of a balloon filled with air. Once squeezed, the air has to go somewhere. In qeneral equilibrium, the somewhere is the stock market. The correlation structure in the proposed model allows this transformation of the state variable into the stock market volatility. In other words, a volatile state variable in the utility function becomes absorbed into the stock market and makes it volatile.

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with the negative correlation structure in the proposed model, this effect becomes magnified.

11) Empirical studies show that real GNP about trend follows a near unit root AR (2) process (Nelson and Plosser (1982), Campbell and Mankiw (1987), etc).

12) Concave with respect to consumption

Since the state variable in the preference covaries with the stock market, the consumption process doesn't have to move together with the stock market J consumption becomes smoother, and the stock market becomes more volatile.

The high equity premium is also justified in terms of the proposed model . when the stock market becomes more volatile (i.e., stock becomes more risky), the safe asset (i.e., the risk-free bond in the model) becomes more valuable since the safe asset provides a hedging service. When the safe asset becomes more valuable, the real interest rate becomes lower. The empirically observed high equity premium can now be explained by this low real interest rate[13)]

## IV. REMAINING ISSUES

There are several remaining issues that need to be discussed. First, we have identified the state variable in the preference structure in terms of its correlation structure with the market portfolio cum business cycle. In the proposed model , therefore, people behave cyclically. But, how can we interpret this state variable in the preference structure? We argue that it is due to a liquidity-related imperfection.[14)]

Many economists have argued that liquidity-related imperfections follow the business cycle, and the failure of the standard model comes from this liquidity-related imperfection (Flavin, 1986; Zeldes, 1988, among others). Moreover , Mehra and Prescott (1988) argue that an alternative preference structure must be shown useful for organiz ing and interpreting not only behaviors in the financial

13) Equity premium is the difference between return on risky stock and return on risk-free bond. The data also show that historical high equity premium comes because of low real interest rate (i.e., low return on risk-free bond).

14) Despite its popularity, term liquidity has not yet gained a precise and unique definition. Nevertheless, there are two poplalar definitions of liquidity which seem more relevant here. One is credit rationing, and another is a divergence between borrowing and lending rates. In both cases, people place a higher value on what they have at present; say, their disposable income. Without this kind of liquidity related imperfection, people are not concerned about their disposable income, but only their permanent income.

markets but also those in other market.[15] If we interpret this alternative preference structure as something related to liquidity, then the extension to other markets follows easily. Liquidity-related imperfection are well recognized in terms of other markets, and many economists argue that economic phenomena can be better explained by incorporating this liquidity-related imperfection.[16]

Indeed, the state dependency of the proposed model can be thought of as a Becker (1965)/Lancaster (1966)/Lucas (1975)[17] taste shock to a utility function, which could, in turn, approximate a decision problem in the face of liquidity constraints. Liquidity constraints are difficult to deal with since they imply multi-linear or non-linear budget constraints. According to the Becker/Lancaster/Lucas Household production Approach, however, we can effectively deal with state dependent preference structure with a single linear budget constraint, by interpreting the state variable as something related to liquidity, say, a liquidity need. The proposed model, therefore, can be thought of as an attempt to capture the flavor of liquidity constraints, and yet avoid the analytical intractability of such a model while working within the context of the standard modelling framework.

The second issue relates to an empirical tests. If this state variable is something that we cannot observe, how can we test this model? The generalized method of moments (GMM) instrumental variables approach might not be applicable in this case, since the GMM approach entails the implicit assumption that the first-order conditions are "exact" in the sense that the functional form of

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15) This is because we are dealing with the same set of preference structure being manifested when we observe asset prices, business cycle, growth of the economy, labor market behavior, and so on. This is exactly the criticism that was imposed upon the non-expected utility-based model (for instance, Epstein and Zin, 1988) and the habit formation-based model by constantinides (1990).

16) For example, in business cycle theory, the observed comovements of consumption and income (or the lack there of) can best be explained by examining the role of liquidity as the additional constraint in the agent's decision problem. Liquidity is also used to arque against the Ricardian doctrine of the equivalence of taxes and deficits. In the literature of implicit labor contracts, the assumption is often made that workers are unable to borrow against future earnings. For a survey of literature, see Hayashi (1985).

17) Becker, Lancaster, and Lucas propose the "household production Function (Hpf)" approach where the household obtains utility from some underlying goods that cannot be bought in the market but are instead produced in the household from inputs of market good and leisure time.

the aqent's obj ective function is known to the econometrician and not subject to unobserved random fluctuations. When the objective function has a stochastic component in it (which is the case in the proposed model), the GMM approach may be misleading.

An alternative approach is to specify the complete economic environment, solve for the endogenous decision variables, and then use the simulated Moment Estimation (SME) methodology. But a challenge is how to endogenize the prices from a fully specified model? Under a fully specified environment we don't have analytic solutions (without fairly special exceptions) for the laws of motion of the decision variables. Nevertheless, this difficulty may be circumvented by using the numerical quadrature-based method proposed by Tauchen (1986). After the model is solved for the decision variables, we fix the parameters of the model # and simulate a series of the state variable in the preference structure. This solution algorithm is in turn used in estimation, by searching over parameter values that minimize the discrepancy between moments of real empirical data values and the values simulated from the proposed model. The SME approach will assess the goodness-of-fit of the proposed model and yield consistent estimators, assuming that the specification of the model is correct.

# APPENDIX 1

With equation (8) and equation (9), we have

$$\begin{aligned} &V_W = e^{-\theta N}W^{-\delta},\ V_{WW} = -\delta e^{-\theta N}W^{-\delta-1},\\ &V_N = -\theta e^{-\theta N}\frac{W^{1-\delta}}{1-\delta},\ V_{NN} = \theta^2 e^{-\theta N}\frac{W^{1-\delta}}{1-\delta},\\ &V_{WN} = -\theta e^{-\theta N}W^{-\delta},\ C^* = e^{\frac{\theta-\lambda}{\delta}N}W,\\ &U(C^*,N) = e^{-\lambda N}\left(\frac{1}{1-\delta}\right)W^{1-\delta}e^{(\theta-\lambda)\left(\frac{1-\delta}{\delta}\right)N}. \end{aligned} \tag{A1}$$

Substituting the three equations, (5), (6), and (7) into (4) yields

$$\rho V = U(C^*,N) + V_W W\left(r+\frac{1}{2}a\right) - V_W C^* + V_N f + \frac{1}{2}V_{NN}g^2 + \frac{1}{2}V_{WN}Wsg\phi \tag{A2}$$

Substituting (Al) into (A2), it follows that

$$\begin{aligned} \rho e^{-\theta N}\frac{W^{1-\delta}}{1-\delta} = \frac{1}{1-\delta}W^{1-\delta}e^{(\theta-\lambda)\left(\frac{1-\delta}{\delta}\right)N-\lambda N} + \left(r+\frac{1}{2}a\right)e^{-\theta N}W^{1-\delta}\\ - e^{\frac{\theta-\lambda}{\delta}N}e^{-\theta N}W^{1-\delta} - f\theta e^{-\theta N}\frac{W^{1-\delta}}{1-\delta} + \frac{1}{2}g^2\theta^2 e^{-\theta N}\frac{W^{1-\delta}}{1-\delta}\\ - \frac{1}{2}sg\phi\theta e^{-\theta N}W^{1-\delta} \end{aligned} \tag{A3}$$

which can be written as

$$\begin{aligned} \left(r+\frac{1}{2}a\right) - \frac{1}{2}\theta sg\phi + \left(\frac{\delta}{1-\delta}\right)e^{\left(\frac{\theta-\lambda}{\delta}\right)N} - \frac{\rho}{1-\delta}\\ - \frac{\theta f}{1-\delta} + \frac{1}{2}\frac{\theta^2 g^2}{1-\delta} = 0 \end{aligned} \tag{A4}$$

Differentiating equation (A4) with respect to N, we have the following equation:

$$a_N - s_N\theta g\phi + 2r_N = 2\left(\frac{\delta}{\delta-1}\right)\left(\frac{\theta-\lambda}{\delta}\right)e^{\left(\frac{\theta-\lambda}{\delta}N\right)} \tag{A5}$$

Since $s_N$ and $r_N$ are assumed to be equal to 0 (Seed footnote 8), equation (A5) directly implies equation (13).

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