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CONNECTING ORBITS FOR SECOND ORDER HAMILTONIAN SYSTEMS

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0. Introduction

This paper concerns with the existence of some kind of connecting orbits for second order Hamiltonian systems of the form

$$\text{(HS)} \qquad q'' + V'(q) = 0.$$

It will be assumed that V has a global maximum,e.g. at $x = 0$. Therefore $q \equiv 0$ is a solution of (HS). We are interested in nontrivial solutions of (HS) that terminate at $x = 0$, i.e.

$$\lim_{t \to \infty} q(t) \equiv q(\infty) = 0 = q'(\infty).$$

Let Ω be a bounded neighborhood of 0 in $\mathbf{R}^n$ and $V \in C^1(\overline{\Omega}, \mathbf{R})$ with $V(x) < V(0)$ for all $x \in \overline{\Omega} \setminus \{0\}$. Under these hypotheses, P.H. Rabinowitz and T. Tanaka [RT] proved the existence of a solution q of (HS) such that $q(0) \in \partial\Omega, q(\infty) = 0 = q'(\infty)$, and $q(t) \in \Omega$ for all $t \in (0, \infty)$. In this paper we are interested in the starting point $q(0) \in \partial\Omega$ of q.

1. Existence Results

Let $\mathbf{R}^+ = [0, \infty]$ and

$$E = \{q \in W^{1,2}_{loc}(\mathbf{R}^+, \mathbf{R}^n) \; ; \int_0^\infty |q'|^2 dt < +\infty \}.$$

E is a Hilbert space under the norm

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$$\|q\|^2 = \int_0^\infty |q'|^2 dt + |q(0)|^2$$

and $E \subset \mathcal{C}(\mathbf{R}^+, \mathbf{R}^n)$. Let $B_\rho(\xi)$ denote the open ball of radius ρ about $\xi \in \mathbf{R}^n$. If $\xi = 0$, we simply write B_ρ.

LEMMA[RAB2]. *Let Ω be a bounded open neighborhood of $0 \in \mathbf{R}^n$. Let $\rho > 0$ be such that $\overline{B}_\rho \subset \Omega$. Set*

$$\beta(\rho) = Min_{x \in \overline{\Omega} \setminus B_\rho} - V(x).$$

Suppose $w \in E$ and $w(t) \in \overline{\Omega} \setminus B_\rho$ for $t \in \cup_{j=1}^k [r_j, s_j]$. Then

$$I(w) \geq \sqrt{2\beta(\rho)} \sum_{j=1}^k |w(r_j) - w(s_j)|.$$

Here $I(q) = \int_0^\infty (\frac{1}{2}\ |q'|^2 - V(q)\ )\, dt.$

Proof.

$$I(w) = \frac{1}{2} \int_0^\infty |w'|^2\, dt - \int_0^\infty V(w)\, dt$$
$$\geq \sum_{j=1}^k (\frac{1}{2} \int_{r_j}^{s_j} |w'|^2\, dt - \int_{r_j}^{s_j} V(w)\, dt).$$

Note that

$$|w(s_j) - w(r_j)|^2 = | \int_{r_j}^{s_j} w'\, dt|^2$$
$$\leq (s_j - r_j) \int_{r_j}^{s_j} |w'|^2\, dt.$$

Hence

$$I(w) \geq \sum_{j=1}^k (\frac{1}{2} \frac{|w(s_j) - w(r_j)|^2}{(s_j - r_j)} + \beta(\rho)(s_j - r_j)\ )$$
$$\geq \sqrt{2\beta(\rho)} \sum_{j=1}^k |w(r_j) - w(s_j)|. \qquad \square$$

THEOREM[RT]. *Let Ω be a bounded neighborhood of 0 in $\mathbf{R}^n$ and $V \in C^1(\overline{\Omega}, \mathbf{R})$ with $V(x) < V(0)$ for all $x \in \overline{\Omega} \setminus \{0\}$. Then there exists a solution q of (HS) such that $q(0) \in \partial\Omega, q(\infty) = 0 = q'(\infty)$, and $q(t) \in \Omega$ for all $t \in (0, \infty)$.*

We now state and prove the main theorem of this paper.

THEOREM. *Let Ω be a bounded neighborhood of 0 in $\mathbf{R}^n$ and $V \in C^1(\overline{\Omega}, \mathbf{R})$ with $V(x) < V(0)$ for all $x \in \overline{\Omega} \setminus \{0\}$. Let $p \in \partial\Omega$ be a point such that*

(1) $\overline{B}_r(0) \cap \partial\Omega = \{p\}$, *where* $\|p\| = r$.

(2) *There is a number* $R > r$ *such that* $(4r^2 + 2\alpha(r))\sqrt{2\beta(r,R)} < R - r$, *where*

$$\alpha(r) = Max_{\|x\| \le r} - V(x),$$
$$\beta(r,R) = Min_{(r+R)/2 \le \|x\| \le R} - V(x).$$

(3) $\Omega^c \cap \{x; \|x\| < R\}$ *is convex.*

Then (HS) has a solution q with $q(0) \in \partial\Omega$ and $\|q(0) - p\| \le \sqrt{R^2 - r^2}$.

Proof. Let Γ be a subset of E defined by

$$\Gamma = \{q \in E; q(0) = p \in \partial\Omega, q(\infty) = 0\},$$

and $q(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}^+$.
For $q \in \Gamma$, consider the functional

$$I(q) = \int_0^\infty (\frac{1}{2}\ |q'|^2 - V(q))dt.$$

Set

$$(*) \qquad c = \inf_{q \in \Gamma} I(q).$$

Let (q_m) be minimizing sequence of $(*)$. Since $V \le 0$ and $\overline{\Omega}$ is compact, the form of I shows that (q_m) is bounded in E. Hence a subsequence of (q_m) converges weakly in E and strongly in $L^\infty_{loc}(\mathbf{R}^+, \mathbf{R}^n)$ to $q \in E$ and $q(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}^+$. Since I is weakly lower semicontinuous,we have $I(q) \le \inf_{w \in \Gamma} I(w)$. Hence $q \in \Gamma$ via Lemma.

Suppose there is a number t_1 such that $q(t_1) \in \partial\Omega, \|q(t_1)\| > R$. Let t_2 be a largest number such that $\|q(t_2)\| = r$.

Let

$$\tilde{q}(t) = \begin{cases} (1-t)p + tq(t_2), & 0 \le t \le 1, \\ q(t+t_2-1), 1 \le t \end{cases}$$

Then

$$\begin{aligned} I(\tilde{q}) &= \int_0^\infty \frac{1}{2}\, |\tilde{q}'|^2 dt - \int_0^\infty V(\tilde{q})dt \\ &\int_0^1 \|q(t_2) - p\|^2 dt - \int_0^\infty V(\tilde{q})dt + \int_1^\infty (\frac{1}{2}\, |\tilde{q}'(t)|^2 - V(\tilde{q}))dt \\ &\le \frac{1}{2}\, 4r^2 + \alpha(r) + \int_{t_2}^\infty \frac{1}{2}\, |q'|^2 - V(q))dt. \end{aligned}$$

On the other hand

$$\begin{aligned} I(q) &= \int_0^\infty (\frac{1}{2}\, |q'|^2 - V(q))dt \\ &= \int_0^{t_2} (\frac{1}{2}\, |q'|^2 - V(q))dt + \int_{t_2}^\infty (\frac{1}{2}\, |q'|^2 - V(q)dt \\ &\ge \sqrt{2\beta(r,R)}(R-r)/2 + \int_{t_2}^\infty (\frac{1}{2}\, |q'|^2 - V(q))dt. \end{aligned}$$

Hence $I(\tilde{q}) < I(q)$, a contradiction. Therefore there is a solution w of (HS) such that $\|w(0)\| \le R$. $\square$

COROLLARY. *Under the hypotheses of the above Theorem , assume further that p is an isolated point. Then there is a solution of (HS) starting at the point p.*

References

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