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Duopoly Model of a Congested Market

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Abstract

A duopoly model is developed in order to examine the effect of imperfect competition on the price-setting behavior of competing providers in a congested market. Multiple Nash price equilibria are found and the implications of such multiple price equilibria are discussed.

1. Introduction

The subject of this paper is to examine the effect of imperfect competition on the price-setting behavior of competing providers in a market with congestion.

Congestion is one of the most important phenomena observed in many markets, such as a telecommunication services market. Congestion occurs whenever users interfere with each other while competing for scarce resources. In such a market, the user time required to consume one unit of the service is increased by the presence of other users on the same system. The user's own time costs involved in consuming the service[2, 9] are typically greater than the price of such telecommunication services as telephone service or electronic message service.

Most of previous studies have focused on the control of congestion through the pricing of services in nonmarket situations, such as highways or airports[1, 3]. An extensive literature review appears in Agnew[1]. Diamond[4] examines the problem of pricing congested facilities in the presence of consumption externalities. Dunn, Eric, and Lecaros[5] present an equilibrium theory for markets in which a number of competing firms offer services subject to congestion in the presence of users

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with the same value of time. In the presence of user heterogeneity with respect to value of time, an equilibrium theory of a congested market is presented by Oh[8]. In a market where providers offer services subject to congestion, the equilibrium analysis is complicated by the fact that congestion is a function of the amount of usage, which in turn depends on user choice.

In order to examine the effect of competition on the price-setting behavior of competing providers, a model of a duopolistic market is developed. The model is used to study the problem of pricing congested services in the presence of imperfect competition. Finally, the existence of multiple price equilibria is illustrated with a numerical example.

2. The Model

Consider a market in which two competing firms offer services subject to congestion that are used by users with different time values. Users are assumed to be clustered into two well-defined groups with different time values. User groups are indexed in increasing order of user time value. The aggregate user demand of each user group is assumed to be the linear demand function

$$q^k = a_k - b_k u^k, \qquad k = 1, 2$$

where a_k and b_k are positive constants, u^k is the user cost of user group k. The user cost of user group k for service j, u_j^k, is defined to be the sum of unit price, p_j, and time cost which is the product of user time value, w^k and unit congestion costs, τ_j. Unit congestion costs are assumed to be linear functions of the amount of services provided, x_j, of the form

$$\tau_j = \alpha_j x_j, \qquad j = 1,2$$

where the congestion coefficient, α_j, is positive.

Then

$$u_j^k = p_j + w^k \alpha_j x_j, \qquad k = 1,2; \quad j = 1,2.$$

It is shown that if u^k is the user cost, then u^k is the minimum user cost[8]. That is

$$u^k = \min_j u_j^k, \qquad k = 1,2; \quad j = 1,2.$$

Given two services and two user groups, the market demands for service j, x_j, can be obtained from the user demand of each user group, q^k[6, 8].

Having derived the market demand for each service as a function of prices, p_1 and p_2, we can now state the decision problems of two firms. In order to make exposition simple and specific, consider the case where users with higher time values consume a higher priced service and users with lower time values consume both a lower priced service and a higher priced service. The derivations of the market demand functions, x_1 and x_2, are contained in Appendix.

Now suppose that firm 1 charges a lower price, p_1, than p_2, the demand faced by the first firm would be the demand for a lower priced (more congested) service. In this case, the first firm solves

$$\max_{p_1} \quad \pi_1 = p_1x_1 - c_1x_1$$

s.t.

$$x_1 = (1+\frac{\alpha_1}{\alpha_2}+b_1w^1\alpha_1+b_2w^2\alpha_1)^{-1}$$

$$\cdot [(a_1+a_2) - \frac{1+b_1w^1\alpha_2+b_2w^2\alpha_2}{w^1\alpha_2}$$

$$\cdot p_1 + \frac{1+b_2\alpha_2(w^2-w^1)}{w^1\alpha_2} \cdot p_2]$$

$$p_1 \leq p_2$$

The reaction function can be obtained by solving the first order conditions for profit maximization. Solving for the first order conditions in terms of p_1 gives

$$p_1 = \frac{1+b_2\alpha_2(w^2-w^1)}{2(1+b_1w^1\alpha_2+b_2w^2\alpha_2)} \cdot p_2 + \frac{w^1\alpha_2(a_1+a_2)}{2(1+b_1w^1\alpha_2+b_2w^2\alpha_2)} + \frac{c_1}{2}. \qquad (1)$$

Consider the decision problem of firm 2. Suppose that firm 2 sets its price, p_2, lower than p_1. The resulting demand faced by firm 2 would be the market demand for a higher priced (less congested) service. The optimization problem of firm 2 is thus

$$\max_{p_2} \quad \pi_2 = p_2x_2 - c_2x_2$$

$$\text{s.t.} \quad x_2 = (1+\frac{\alpha_2}{\alpha_1}+b_1w^1\alpha_2+b_2w^2\alpha_2)^{-1}$$

$$\cdot [(a_1+a_2)+\frac{1}{w^1\alpha_1} \cdot p_1 - \frac{1+b_1w^1\alpha_1+b_2w^1\alpha_1}{w^1\alpha_1} \cdot p_2]$$

$$p_2 \geq p_1$$

Solving for the first order conditions in terms of p_2 gives

$$p_2 = \frac{1}{2(1+b_1w^1\alpha_1+b_2w^1\alpha_1)} \cdot p_1 + \frac{w^1\alpha_1(a_1+a_2)}{2(1+b_1w^1\alpha_1+b_2w^1\alpha_1)} + \frac{c_2}{2}. \qquad (2)$$

Next consider the case in which firm 1 sets its price, p_1, higher than p_2. The resulting demand faced by firm 1 would be the market demand for a higher priced(less congested) service. The first firm solves

$$\max_{p_1} \quad \pi_1 = p_1x_1 - c_1x_1$$

$$\text{s.t.} \quad x_1 = (1+\frac{\alpha_1}{\alpha_2}+b_1w^1\alpha_1+b_2w^2\alpha_1)^{-1}$$

$$\cdot [(a_1+a_2) - \frac{1+b_1w^1\alpha_2+b_2w^2\alpha_2}{w^1\alpha_2} \cdot p_1 + \frac{1}{w^1\alpha_2} \cdot p_2]$$

$$p_1 \geq p_2$$

Solving for the first order conditions in terms of p_1 gives

$$p_1 = \frac{1}{2(1+b_1w^1\alpha_2+b_2w^1\alpha_2)} \cdot p_2$$

$$+\frac{w^1\alpha_2\,(a_1+a_2)}{2(1+b_1w^1\alpha_2+b_2w^1\alpha_2)}+\frac{c_1}{2}. \qquad (3)$$

Suppose that firm 2 sets its price, p_2, lower than p_1. The demand faced by firm 2 would be the market demand for a lower priced(more congested) service. The decision problem of firm 2 is thus

$$\max_{p_2} \qquad \pi_2 = p_2x_2 - c_2x_2$$

$$s.t. \qquad x_2 = (1+\frac{\alpha_2}{\alpha_1}+b_1w^1\alpha_2+b_2w^2\alpha_2)^{-1}$$

$$\cdot\,[(a_1+a_2)+\frac{1+b_2\alpha_1(w^2-w^1)}{w^1\alpha_1}\cdot p_1$$

$$-\frac{1+b_1w^1\alpha_1+b_2w^2\alpha_1}{w^1\alpha_1}\cdot p_2]$$

$$p_2 \leq p_1$$

Solving for the first order conditions in terms of p_2 gives

$$p_2 = \frac{1+b_2\alpha_1\,(w^2-w^1)}{2(1+b_1w^1\alpha_1+b_2w^2\alpha_1)}\cdot p_1$$

$$+\frac{w^1\alpha_1\,(a_1+a_2)}{2(1+b_1w^1\alpha_1+b_2w^2\alpha_1)}+\frac{c_2}{2}. \qquad (4)$$

Given the market demand faced by each firm, the noncooperative price equilibrium can be obtained by maximizing π_1 with respect to p_1 and π_2 with respect to p_2[7].

3. Derivation of Nash Price Equilibrium

Each firm is assumed to set the price for its service so as to maximize its profit, assuming that his competitor's price remains constant. Under this behavioral assumption, the following two different equilibria can exist.

Case A : Firm 1 sets its price, p_1, lower than p_2 and firm 2 charges a higher price, p_2, than p_1

Solving for p_1 and p_2 in Eqs. (1) and (2) yields the equilibrium prices, p_1^A and p_2^A.

Case B : Firm 2 sets its price, p_2, lower than p_1 and firm 1 charges a higher price, p_1, than p_2.

In this case, solving for p_1 and p_2 in Eqs. (3) and (4) yields another equilibrium prices, p_1^B and p_2^B.

Having obtained two different price equilibria, we consider their implications. In general, two firms may get different profits at the different equilibrium. This implies that both firms have an incentive to change their prices in order to improve their profits. Therefore, the instability of a price equilibrium exists in a congested market with inhomogeneous user groups with respect to value of time.

4. A Numerical Example

The existence of multiple price equilibria can be best illustrated with a simple numerical example. It is assumed that a user population is divided into two user groups, each of which has different time values. In this case, let $w^1 = 1$ and $w^2 = 2$. The user demands and congestion functions are assumed to be linear as follows :

$$q^1 = 10 - u^1$$
$$q^2 = 10 - u^2$$
$$\tau_1(x_1) = \alpha_1 x_1$$
$$\tau_2(x_2) = \alpha_2 x_2$$

where α_j denotes the congestion coefficient of service j and is assumed to be $\alpha_1 = 1$ and $\alpha_2 = 2$.

The unit cost of providing each service is assumed to be zero. Suppose that firm 1 sets its price, p_1, lower than p_2 and firm 2 sets its price, p_2, higher than p_1. In this case, the market demands, x_1 and x_2, are

$$x_1 = \frac{1}{4.5}(20 - 3.5p_1 + 1.5p_2)$$

$$x_2 = \frac{1}{9}(20 + p_1 - 3p_2)$$

The resulting equilibrium prices, market demands, user demands, and profits can be obtained as follows :

$$p_1^A = 3.7 \qquad \pi_1^A = 10.66$$
$$p_2^A = 3.95 \qquad \pi_2^A = 5.2$$
$$x_1^A = 2.88 \qquad q_1^A = 3.42$$
$$x_2^A = 1.32 \qquad q_2^A = 0.78$$

At equilibrium A, user group 1 consumes both services, and user group 2 consumes high-priced services only.

Another price equilibrium B can be obtained when firm 1 charges a higher price, p_1, than p_2. In this case, the market demands, x_1 and x_2, are

$$x_1 = \frac{1}{4.5}(20 - 2.5p_1 + 0.5p_2)$$

$$x_2 = \frac{1}{9}(20 + 2p_1 - 4p_2)$$

The equilibrium prices, market demands, user demands, and profits are

$$p_1^B = 4.36 \qquad \pi_1^B = 10.55$$
$$p_2^B = 3.58 \qquad \pi_2^B = 5.72$$
$$x_1^B = 2.42 \qquad q_1^B = 3.22$$
$$x_2^B = 1.6 \qquad q_2^B = 0.8$$

Thus, two different price equilibria, A and B, have been found. Firm 1 earns a higher profit at p_1^A than p_1^B, and firm 2 earns a higher profit at p_2^B than p_2^A. At equilibrium A, firm 2 has an incentive to move to equilibrium B. However, firm 1 does not have an incentive to move to the equilibrium B, because firm 1's payoff at B is lower than its payoff at A. We may argue that each price equilibrium is locally stable but globally unstable.

5. Conclusions

In order to examine the effect of competition on the price-setting behavior of competing providers, a model of a duopolistic market has been developed. The basic behavioral assumption used in this paper is that of Nash, since we are interested in a noncooperative price equilibrium. Given this behavioral assumption, it has been shown that multiple price equilibria can exist in a market with congestion. The existence of multiple price equilibria is seen as a result of user heterogeneity with respect to value of time.

The implications of such multiple price equilibria may be summarized as follows. In general, providers may get different profits at different equilibrium. This implies that providers have an incentive to change their prices in order to improve their profits. Therefore, the instability of the price equilibrium exists in a congested market with inhomogeneous user groups with respect to value of time. Two other possibilities, not analyzed in this paper are that both providers may collude or merge, because each provider may threaten the rival by changing his price, and that the unstable price competition may result in nonprice competition.

Appendix

This appendix contains the derivation of the market demand for each service as a function of prices, p_1 and p_2. Consider only the demand equilibrium case in which users with higher time values consume a higher priced service and users with lower time values consume both a lower priced service and a higher priced service. Given two user groups and two competing firms, two cases are of interest.

Case A : Suppose that firm 1 sets its price, p_1, lower than p_2 and firm 2 charges a higher price, p_2, than p_1.

In this case, user group 1 incurs the same user cost for both services and user group 2 incurs a least user cost for service 2. The corresponding demand equilibrium conditions are

$$u^1 = p_1 + w^1 \alpha_1 x_1 = p_2 + w^1 \alpha_2 x_2 \cdots \text{(A1)}$$

$$u^2 = p_2 + w^2 \alpha_2 x_2 \leq p_1 + w^2 \alpha_1 x_1 \cdots \text{(A2)}$$

$$x_1 + x_2 = \sum_{k=1}^{2} (a_k - b_k u^k). \cdots\cdots\cdots\cdots \text{(A3)}$$

Substituting u^1 and u^2 from Eqs. (A1) and (A2) into Eq. (A3) and using the relationship between x_1 and x_2 from Eq. (A1), we obtain the market demand functions, x_1 and x_2 in terms of prices, p_1 and p_2 :

$$x_1 = \left(1 + \frac{\alpha_1}{\alpha_2} + b_1 w^1 \alpha_1 + b_2 w^2 \alpha_1\right)^{-1}$$

$$\cdot \left[(a_1 + a_2) - \frac{1 + b_1 w^1 \alpha_2 + b_2 w^2 \alpha_2}{w^1 \alpha_2} \cdot p_1 \right.$$

$$+\frac{1+b_2\alpha_2(w^2-w^1)}{w^1\alpha_2}\cdot p_2] \quad \text{(A4)}$$

$$x_2=(1+\frac{\alpha_2}{\alpha_1}+b_1w^1\alpha_2+b_2w^2\alpha_2)^{-1}$$

$$\cdot[(a_1+a_2)+\frac{1}{w^1\alpha_1}\cdot p_1$$

$$-\frac{1+b_1w^1\alpha_1+b_2w^1\alpha_1}{w^1\alpha_1}\cdot p_2]. \quad \text{(A5)}$$

Case B : Suppose that firm 2 sets its price, p_2, lower than p_1 and firm 1 charges a higher price, p_1, than p_2.

In this case, user group 1 incurs the same user cost for both services and user group 2 incurs a minimum user cost for service 1. The corresponding demand equilibrium conditions are

$$u^1 = p_2 + w^1\alpha_2 x_2 = p_1 + w^1\alpha_1 x_1 \cdots \text{(A6)}$$

$$u^2 = p_1 + w^2\alpha_1 x_1 \leq p_2 + w^2\alpha_2 x_2 \cdots \text{(A7)}$$

$$x_1 + x_2 = \sum_{k=1}^{2}(a_k - b_k u^k) \cdots\cdots\cdots \text{(A8)}$$

The market demand functions, x_1 and x_2 can be obtained from Eqs. (A6), (A7) and (A8):

$$x_1=(1+\frac{\alpha_1}{\alpha_2}+b_1w^1\alpha_1+b_2w^2\alpha_1)^{-1}$$

$$\cdot[(a_1+a_2)-\frac{1+b_1w^1\alpha_2+b_2w^2\alpha_2}{w^1\alpha_2}\cdot p_1$$

$$+\frac{1}{w^1\alpha_2}\cdot p_2] \quad \text{(A9)}$$

$$x_2=(1+\frac{\alpha_2}{\alpha_1}+b_1w^1\alpha_2+b_2w^2\alpha_2)^{-1}$$

$$\cdot[(a_1+a_2)+\frac{1+b_2\alpha_1(w^2-w^1)}{w^1\alpha_1}\cdot p_1$$

$$-\frac{1+b_1w^1\alpha_1+b_2w^2\alpha_1}{w^1\alpha_1}\cdot p_2] \quad \text{(A10)}$$

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