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RELATIONSHIP BETWEEN QUEUE LENGTHS AND WAITING TIMES FOR QUEUEING NETWORK MODELS

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Abstract

For general open queueing network models, the relationship between weak limits of queue lengths and waiting times at stations is investigated under heavy traffic situations. It is shown that under suitable normalization, weak convergence of queue lengths and arrival processes is a sufficient condition for that of waiting times, and is also necessary condition when the network is of feedforward type. Moreover, these weak limits for queue lengths and waiting times are shown to be simply related.

1. introduction

In this paper we investigate, in a setting general as much as possible, the relationship between weak limits of the queue length and waiting time of queueing systems under heavy traffic situations. To explain more precisely what we do in this paper, let us consider the queueing model GI/G/1/∞(i.e.,interarrival and service times are independent sequences of i.i.d. random variables). We think of a sequence of such models, and let $Q_n(t)$ be the queue length at time t and $W_n(k)$ be the waiting time of kth customer for the nth queue model. It is well known that under a heavy traffic condition, normalized processes $\hat{Q}_n(t)=(1/\sqrt{n})Q_n(nt)$ and $\hat{W}_n(t)=(1/\sqrt{n})W_n([nt])$ converge in law to reflecting Brownian motions $\hat{Q}(t)$ and $\hat{W}(t)$ respectively as $n\to\infty$[8]. There are various ways for obtaining this result, and if we look at closely these proofs, we see that the existence of weak limit of $\{\hat{Q}_n;n\geq 1\}$ is necessary and

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sufficient for that of $\{\hat{W}_n; n \geq 1\}$, and the weak limits $\hat{Q}$ of $\{\hat{Q}_n; n \geq 1\}$ and $\hat{W}$ of $\{\hat{W}_n; n \geq 1\}$ are simply related, i.e.,the law of $\hat{Q}(\cdot)$ is equivalent to that of $\lambda\hat{W}(\lambda\cdot)$ for suitable positive constant λ. However as long as this last result is concerned, we can prove it under much more general conditions. In fact, the following fact will be shown: We assume only that (1) $Q_n(t)$ is expressed by $Q_n(t) = A_n(t) - D_n(t)$ where $A_n(t)$ and $D_n(t)$ are counting processes which count arrival and departure customers (for simplicity $Q_n(0) = 0$ is assumed), (2) the waiting time $W_n(k)$ for the k th customer is defined as $W_n(k) = D_n^{-1}(k) - A_n^{-1}(k)$ where $A_n^{-1}(\cdot)$ and $D_n^{-1}(\cdot)$ are inverse processes of $A_n(t)$ and $D_n(t)$ respectively. Then for a suitably normalized process $\hat{A}_n(t)$ of $A_n(t)$, the joint convergence $\{\hat{Q}_n, \hat{A}_n; n \geq 1\}$ is necessary and sufficient for that of $\{\hat{W}_n, \hat{A}_n; n \geq 1\}$. Moreover, the weak limits of $\{\hat{Q}_n; n \geq 1\}$ and $\{\hat{W}_n; n \geq 1\}$ are simply related as was stated before.

In this paper we discuss the queueing networks of Markovian type, and obtain the similar result as above. A major drawback of our result is, though our model is quite general, in the definition of waiting time (see (3.4) is Section 3); it lose the meaning as waiting time for some systems such as queue models with multiple severs at a station.

The central limit theorem version of Little's formula (the relationship between queue lengths and waiting times) was discussed in [1],[2]. The other topics related to our paper are studied in [5] and [6].

We denote by $D([0,\infty), R^d)$ the space of function $f:[0,\infty) \to R^d$ which are right continuous and left limits, and we endow this space with the Skorohod topology [3]. We also denote by $\overset{P}{\to}$ and $\overset{L}{\to}$ the convergence in probability and in law, respectively.

The rest of this paper is organized as follows. In Section 2, we describe the structure of the queueing network which we treat in this paper. In Section 3, we derive the relationship between weak limits of queue lengths and waiting times at stations. Finally,in Section 4, we illustrate an example.

2. Model description

The queueing network consists of K stations, and we assume that the customer routing within the network is Markovian. Let P be the routing matrix for the network. That is, P_{ij} represents the probability that a customer completing service at station i goes immediately to station j ; $1 - \sum_{j=1}^{K} P_{ij}$ is the probability that a customer released from station i leaves the network. Let $A_i(t), 1 \leq i \leq K$, be the number of customers who arrive at the ith station by time t from the outside of the network , and $D_i(t), 1 \leq i \leq K$, be the number of customers released from station i by time t. Suppose that $R_i(m), m = 1, 2, \cdots$, is a random vector which route the mth customer completing service at the ith station. More precisely , $R_i(m) = e_j$ (e_j is the jth unit vector) if and ony if the mth customer to be served at the i th station is routed immediately, upon completing service at station i, to station j, and $P(R_i(m) = e_j) = P_{ij}$. Let $Q(t) = (Q_1(t), \cdots, Q_K(t))$ be the vector whose ith component $Q_{i(t)}$ is the number of customers at the ith station at time t. We also let $D(t) = (D_1(t), \cdots, D_K(t))$. Our basic assumption is that $Q(t)$ is represented by:

$$Q(t) = Q(0) + A(t) + \sum_{i=1}^{K} \sum_{m=1}^{D_i(t)} R_i(m) - D(t). \qquad (2.1)$$

Note that $\sum_{m=1}^{D_i(t)} R_i(m)$ is a vector in which the kth component equals the number of customer routed from station i to K over $[0,t]$.

3. Basic results

We consider now a sequence of queueing networks of the type described in Section 2, indexed by $n \geq 1$. Let (Ω_n, F_n, P_n) be the probability space on which processes of the n th such network are defined. All the notations established in Section 2 are carried forward, except that we append an n in a convenient place to denote a quantity which depends on n. We assume that K and the routing matrix are independent of n. Thus queue length vector $Q_n(t)$ for the nth network is represented by:

$$Q_n(t) = Q_n(0) + A_n(t) + \sum_{i=1}^{K} \sum_{m=1}^{D_n^i(t)} R_i(m) - D_n(t), \quad (3.1)$$

and to simplify our discussion, we assume $Q_n(0) = 0$ for $n \geq 1$. We make the following assumptions:

Assumption 1. $(I-P)^{-1}$ exists where I is $K \times K$ identity matrix.

Assumption 2. There exists a sequence of nonnegative constant vector $\lambda_n, n \geq 1$, such that $\lambda_n \to \lambda$ and

$$V_n(t) \equiv \left(\frac{1}{\sqrt{n}} (A_n(nt) - \lambda_n nt), \frac{1}{\sqrt{n}} \sum_{m=1}^{[nt]} (R_i(m) - P_i), i = 1, \cdots, K \right) \quad (3.2)$$

converges weakly, as $n \to \infty$, to a continuous process $V(t) = (\hat{A}(t), \hat{R}_i(t), i = 1, \cdots, K)$ in $D([0,\infty), R^{K+K^2})$ where P_i is the ith row vector of P. Moreover, if we let $a_i = (\lambda(I-P)^{-1})_i$, the ith element of $\lambda(I-P)^{-1}, 1 \leq i \leq K$, then $a_i > 0, 1 \leq i \leq K$.

Assumption 3. Let

$$A_n(t) \equiv \left(\frac{1}{\sqrt{n}} Q_n(nt), \frac{1}{\sqrt{n}} (A_n(nt) - \lambda_n nt), \frac{1}{\sqrt{n}} \sum_{m=1}^{[nt]} (R_i(m) - P_i) , i = 1, \cdots, K \right), n \geq 1 \quad (3.3)$$

Then $A_n(t)$ converges weakly, as $n \to \infty$, to a continuous process $A(t) = (\hat{Q}(t), \hat{A}(t), \hat{R}_i(t), i = 1, \cdots, K)$ in $D([0,\infty), R^v)$ where $v = 2K + K^2$.

Let $W_n^i(k)$ be the waiting time at station i of a customer whose arrival order at station i is kth. More precisely, it will be defined as follows. For an arbitrary cadlag process $X(t)$ which is unbounded above and $X(0) \geq 0$, we define the inverse process $X^{-1}(t)$ by $X^{-1}(t) = \inf\{s; X(s) > t\}$. Then $W_n^i(k)$ is defined by

$$W_n^i(k) = (D_n^i)^{-1}(k) - (\alpha_n^i)^{-1}(k) \quad (3.4)$$

where the process $\alpha_n^i(t)$ is the ith element of the vector process

$$\alpha_n(t) = A_n(t) + \sum_{i=1}^{K} \sum_{m=1}^{D_n^i(t)} R_i(m). \quad (3.5)$$

Let us define processes $\hat{W}_n(t), n \geq 1$, by

$$\hat{W}_n(t) = \left(\frac{1}{\sqrt{n}} W_n^1([nt]), \cdots, \frac{1}{\sqrt{n}} W_n^K([nt]) \right). \quad (3.6)$$

Then our interest is to express the weak limit of $\hat{W}(t)$ (if exists) by the processes $\hat{Q}(t),\hat{A}(t)$ and $\hat{R}_i(t), 1\leq i\leq K$, which appear in Assumption 2 as limit processes. The result is contained in the following Theorem.

Theorem 1. Assume assumptions 1, 2 and 3. Then

$$(A_n(t),\hat{W}_n(t)) \xrightarrow{L} (A(t),\hat{W}(t)) \tag{3.7}$$

in $D([0,\infty),R^{K+v})$. where weak limit $\hat{W}(t)$ is expressed as

$$\hat{W}(t) = \left(\frac{1}{a_1}\hat{Q}_1\left(\frac{t}{a_1}\right),\cdots,\frac{1}{a_K}\hat{Q}_K\left(\frac{t}{a_K}\right)\right) \tag{3.8}$$

Here $a_i, i=1,\cdots,K$ were defined in Assumption 2.

Proof. For the proof of Theorem 1 we need several lemmas.

Lemma 1. Let $X_n, Y_n, n\geq 1$ are real valued cadlag processes which are unbounded above and $X_n(0)\geq 0$ and $Y_n(0)\geq 0$. Suppose, for an arbitrary R^d-valued cadlag processes $Z_n(t), n\geq 1$, that

$$(Z_n(t),\hat{X}_n(t),\hat{Y}_n(t)) \xrightarrow{L} (Z(t),\hat{X}(t),\hat{Y}(t)) \tag{3.9}$$

in $D([0,\infty),R^{d+2})$ where

$$\hat{X}_n(t) = \frac{1}{\sqrt{n}}(X_n(nt)-a_n nt) \tag{3.10}$$

$$\hat{Y}_n(t) = \frac{1}{\sqrt{n}}(Y_n(nt)-b_n nt) \tag{3.11}$$

and $\hat{X},\hat{Y}$ are continuous processes. Then if $a_n\to a\geq 0$ and $b_n\to b\geq 0$,

$$(Z_n(t),\hat{X}_n(t),\hat{Y}_n(t),\overline{X}_n(t),\overline{Y}_n(t)) \xrightarrow{L} \left(Z(t),\hat{X}(t),\hat{Y}(t),-\frac{1}{a}\hat{X}\left(\frac{t}{a}\right),-\frac{1}{b}\hat{Y}\left(\frac{t}{b}\right)\right) \tag{3.12}$$

in $D([0,\infty),R^{d+4})$ where

$$\overline{X}_n(t) = \frac{1}{\sqrt{n}}\left(X_n^{-1}(nt)-\frac{1}{a_n}nt\right) \tag{3.13}$$

$$\overline{Y}_n(t) = \frac{1}{\sqrt{n}}\left(Y_n^{-1}(nt)-\frac{1}{b_n}nt\right) \tag{3.14}$$

Proof. We can write $\hat{X}_n(t)$ and $\hat{Y}_n(t)$ as

$$\hat{X}_n(t) = a_n\sqrt{n}\left(\frac{1}{na_n}X_n(nt)-t\right) \tag{3.15}$$

$$\hat{Y}_n(t) = b_n\sqrt{n}\left(\frac{1}{nb_n}Y_n(nt)-t\right) \tag{3.16}$$

Since inverse processes of $(1/na_n)X_n(nt)$ and $(1/nb_n)Y_n(nt)$ are $(1/n)X_n^{-1}(a_n nt)$ and $(1/n)Y_n^{-1}(b_n nt)$ respectively, by Whitt[9],

$$(Z_n(t),\hat{X}_n(t),\hat{Y}_n(t),U_n(t),V_n(t)) \xrightarrow{L} (Z(t),\hat{X}(t),\hat{Y}(t),-\hat{X}(t),-\hat{Y}(t)) \tag{3.17}$$

in $D([0,\infty),R^{d+4})$ where

$$U_n(t) = a_n\sqrt{n}\left(\frac{1}{n}X_n^{-1}(a_n nt)-t\right) \tag{3.18}$$

$$V_n(t) = b_n\sqrt{n}\left(\frac{1}{n}Y_n^{-1}(b_n nt)-t\right) \tag{3.19}$$

Then the conclusion follows if we note that $\overline{X}_n(t)=(1/a_n)U_n((1/a_n)t)$ and $\overline{Y}_n(t)=(1/b_n)V_n((1/b_n)t)$ and apply a composition theorem in Whitt [9] □

Lemma 2. Let $\psi_n(t)=(1/n)D_n(nt)$ and $\psi(t)=\lambda(I-P)^{-1}t$. Then $sup_{t\in T}|\psi_n(t)-\psi(t)|\overset{P}{\to}0$ as $n\to\infty$, where T is an arbitrary interval of $[0,\infty)$.

Proof. Since $\psi_n(t)$ and $\psi(t)$ are both increasing in t and $\psi(t)$ is continuous and deterministic, it suffices to show that $\psi_n(t)\overset{P}{\to}\psi(t)$ as $n\to\infty$ for each $t\geq 0$. Let

$$G_n^i(t)=\frac{1}{D_n^i(nt)}\sum_{m=1}^{D_n^i(nt)}R_i(m),\qquad 1\leq i\leq K. \tag{3.20}$$

Then from (3.1),

$$\frac{1}{n}Q_n(nt)=\frac{1}{n}A_n(nt)-\psi_n(t)(I-G_n(t)) \tag{3.21}$$

where $G_n(t)=(G_n^1(t),\cdots,G_n^K(t))$. To show our conclusion, we may and do assume that $A_n(t)$ converge to $A(t)$ uniformly on any compact t-interval with probability one (Skorohod theorem). Then we will show that $D_n^i(nt)\to\infty$ with probability one for $t>0$ and for all $i(1\leq i\leq K)$. First suppose that for an i, $\lambda_i>0$. (Note that since $a_i>0$ for all $i(1\leq i\leq K)$ by Assumption 2, there exists such $\lambda_i>0$. Since

$$\psi_n(t)\geq\frac{1}{n}A_n(nt)-\frac{1}{n}Q_n(nt) \tag{3.22}$$

(for vectors a and $b, a\geq b$ implies $a_i\geq b_i$ for each $(i)$and $(1/n)A_n(nt)\to\lambda t$ and $(1/n)Q_n(nt)\to 0$ with probability one, $liminf_{n\to\infty}\psi_n(t)\geq\lambda t$ with probability one. Hence $liminf_{n\to\infty}\psi_n^i(t)\geq\lambda_i t>0$, and this implies that with probability one, $D_n^i(nt)\to\infty$ as $n\to\infty$ for $t>0$ when $\lambda_i>0$. Next let $i(1\leq i\leq K)$ be arbitrary. It is easy show that in the view of Assumption 1 and 2 that $a_i>0$ for all $i(1\leq i\leq K)$, there exist an $m\geq 1$ and $i_0,i_1,\cdots,i_{m-1}$ such that $\lambda_{i_0}>0, P_{i_0i_1}>0, P_{i_1i_2}>0,\cdots, P_{i_{m-1}i}>0$. Since $\lambda_{i_0}>0$, from the above result we have $liminf_{n\to\infty}(1/n)D_n^{i_0}(nt)\geq\lambda_{i_0}t>0$ with probability one for $t>0$. Then, for $t>0$,

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{m=1}^{D_n^{i_0}(nt)}(R_{i_0}(m))_{i_1}\geq\liminf_{n\to\infty}\frac{1}{n}D_n^{i_0}(nt).$$

$$\liminf_{n\to\infty}\frac{1}{D_n^{i_0}(nt)}\sum_{m=1}^{D_n^{i_0}(nt)}(R_{i_0}(m))_{i_1}\geq\lambda_{i_0}P_{i_0i_1}t>0. \tag{3.23}$$

Hence, noting from (3.1) that

$$\frac{1}{n}D_n^{i_1}(nt)\geq\frac{1}{n}\sum_{m=1}^{D_n^{i_0}(nt)}(R_{i_0}(m))_{i_1}-\frac{1}{n}Q_n^{i_1}(nt), \tag{3.24}$$

we have that $\liminf_{n\to\infty}(1/n)D_n^{i_1}(nt)\geq\lambda_{i_0}P_{i_0i_1}t>0$ with probability one for $t>0$. Proceeding similarly as avove. we have that $\liminf_{n\to\infty}(1/n)D_n^{i_2}(nt)\geq\lambda_{i_0}P_{i_0i_1}P_{i_1i_2}t>0$ with probability one for $t>0$, and finally we see that $\liminf_{n\to\infty}(1/n)D_n^{i}(nt)\geq\lambda_{i_0}P_{i_0i_1}\cdots P_{i_{m-1}i}t>0$ for $t>0$. Thus $D_n^i(nt)\to\infty$ with probability one for $t>0$ and for all $i(1\leq i\leq K)$. Then this implies that $G_n(t)\to P$ as $n\to\infty$ for $t>0$ with probability one, and by (3.22), $\psi_n(t)\to\lambda(I-P)^{-1}t(=\psi(t))$ with probability one as $n\to\infty$ fot $t>0$. Thus our conclusion follows. □

Lemma 3. Let

$$\hat{D}_n(t)=\frac{1}{n}(D_n(t)-\lambda_n(I-P)^{-1}nt), n\geq 1. \tag{3.25}$$

Then

$$(A_n(t),\hat{D}_n(t)) \xrightarrow{L} (A(t),\hat{D}(t)) \tag{3.26}$$

in $D([0,\infty),R^{K+v})$, where $\hat{D}(t)$ is defined by

$$\hat{D}(t) = \left[\hat{A}(t) + \sum_{i=1}^{K} \hat{R}_i(\psi_i(t)) - \hat{Q}(t)\right](I-P)^{-1}. \tag{3.27}$$

Proof. From (3.1),

$$D_n(t)(I-P) = A_n(t) + \sum_{i=1}^{K}\sum_{m=1}^{D_n^i(t)} (R_i(m)-P_i) - Q_n(nt). \tag{3.28}$$

Hence $\hat{D}_n(t)$ can be written as

$$\hat{D}_n(t) \equiv \left(\frac{1}{\sqrt{n}}(A_n(nt)-\lambda_n nt) + \frac{1}{\sqrt{n}}\sum_{i=1}^{K} \sum_{m=1}^{[n\psi_n^i(t)]} (R_i(m)-P_i) - \frac{1}{\sqrt{n}}Q_n(nt)\right)(I-P)^{-1} \tag{3.29}$$

where $\psi_n^i(t)$ is the ith element of $\psi_n(t)$. Then we have our conclusion by Lemma 2 and the following general fact: Suppose $(X_n(t), Y_n(t), Z_n(t)) \xrightarrow{L} (X(t),Y(t),Z(t))$ in $D([0,\infty),R^3)$, where Z_n is non-decreasing and Y is continuous. Then

$$(X_n(t),Y_n(t),Z_n(t),X_n(t)+Y_n(Z_n(t))) \xrightarrow{L} (X(t),Y(t),Z(t),X(t)+Y(Z(t))) \tag{3.30}$$

in $D([0,\infty),R^4)$. □

We now prove the Theorem 1. We note that

$$W_n^i(k) = (D_n^i)^{-1}(k) - (D_n^i+Q_n^i)^{-1}(k), 1 \le i \le K, \tag{3.31}$$

where $(D_n^i+Q_n^i)^{-1}(t)$ is the inverse process of $D_n^i(t)+Q_n^i(t)$. By Lemma 3,

$$b_n(t) \equiv \left(A_n(t), \frac{1}{\sqrt{n}}(D_n(nt)-\lambda_n(I-P)^{-1}nt), \frac{1}{\sqrt{n}}(D_n(nt)+Q_n(nt)-\lambda_n(I-P)^{-1}nt)\right) \xrightarrow{L} (A(t),\hat{D}(t),\hat{D}(t)+\hat{Q}(t)) \equiv b(t), \tag{3.32}$$

in $D([0,\infty),R^{2K+v})$ as $n\to\infty$. Thus by Lemma 1

$$\left(b_n(t), \frac{1}{\sqrt{n}}\left((D_n^i)^{-1}(nt) - \frac{1}{a_n^i}nt\right), \frac{1}{\sqrt{n}}\left((D_n^i+Q_n^i)^{-1}(nt) - \frac{1}{a_n^i}nt\right), 1 \le i \le K\right) \xrightarrow{L} \left(b(t), -\frac{1}{a_i}\hat{D}\left(\frac{t}{a_i}\right), -\frac{1}{a_i}\left(\hat{D}\left(\frac{t}{a_i}\right)+\hat{Q}\left(\frac{t}{a_i}\right)\right)\right), 1 \le i \le K \tag{3.33}$$

in $D([0,\infty),R^{4K+v})$ where a_n^i is the ith element of $\lambda_n(I-P)^{-1}$. Now, since

$$W_n^i(t) = \frac{1}{\sqrt{n}}\left((D_n^i)^{-1}([nt]) - (D_n^i+Q_n^i)^{-1}([nt])\right), 1 \le i \le K, \tag{3.34}$$

we reach the conclusion. □

Remark 1. $W_n^i(k)$ defined by (3.4) has the meaning as waiting time (or sojourn time) only when a customer who comes first completes the service first. This is the case when the service descipline is first come, first served base and there exists a single server at sation i. If there are multiple servers at station i and service times are random, $W_n^i(k)$ loses the meaning as waiting times as was remarked in introduction.

Our next concern is to investigate whether the converse of Theorem 1 holds or not. Thus we set the Assumption 4.

Assumption 4. Let

$$B_n(t)=(\frac{1}{\sqrt{n}}W_n(nt),\frac{1}{\sqrt{n}}(A_n(nt)-\lambda_n nt),$$

$$\frac{1}{\sqrt{n}}\sum_{m=1}^{[nt]}(R_i(m)-P_i),i=1,\cdots\cdots,K),n\geq 1 \qquad (3.35)$$

Then $B_n(t)$ converges weakly, as $n\to\infty$, to a continuous process:
$B(t)=(\hat{W}(t),\hat{A}(t),\hat{R}_i(t),i=1,\cdots,K)$ in $D([0,\infty),(R^{v}))$. Under this condition, we have the following theorem

Theorem 2. Assume Assumption 1, 2 and 4. We further suppose that $P_{ij}=0$ if $i\geq j$. Then

$$(B_n(t),\hat{Q}_n(t))\xrightarrow{L}(B(t),\hat{Q}(t)) \qquad (3.36)$$

in $D([0,\infty),R^{K+v})$, where $\hat{Q}_n(t)=(1/\sqrt{n}Q_n^1(nt),\cdots,1/\sqrt{n}Q_n^K(nt))$ and $\hat{Q}(t)$ is given by

$$\hat{Q}(t)=(\hat{Q}_1(t),\cdots,\hat{Q}_K(t)), \qquad (3.37)$$

$$\hat{Q}_i(t)=a_i\hat{W}_i(a_it),1\leq i\leq K. \qquad (3.38)$$

Proof. We prove Theorem 2 inductively. Let $D\{i;1\leq i\leq K,P_{ij}=0$ for $j=1,\cdots,\}$. Then we may assume that each station of the network is numbered so that there exists i_0 satisfying $D\{1,2,\cdots,i_0\}$.

Step 1 (1) We prove that the following (a) and (b) hold:

(a) for each $t\geq 0$, $D_n^i(nt)/n \xrightarrow{P} a_it,1\leq i\leq i_0.$

(b) $(B_n(t),\hat{D}_n^i(t),1\leq i\leq i_0)\xrightarrow{L}$
$(B(t),\tilde{D}_i(t),1\leq i\leq i_0)$ in $D([0,\infty),R^{i_0+v})$,

where $\hat{D}_n^i(t)$ was defined in Lemma 3 and

$$\tilde{D}_i(t)=-\lambda_i\hat{W}_i(\lambda_it)+\hat{A}_i(t),1\leq i\leq i_0.$$

(a) : This is trivial if we note that (i) $Q_n^i(t)=A_n^i(t)-D_n^i(t),1\leq i\leq i_0,$
(ii) $(A_{n_i}^i(nt)/n,Q_n^i(nt)/n)\xrightarrow{P}(\lambda_it,0)$ for each $t\geq 0$, and (iii) $a_n^{i_i}$ the ith element of $\lambda_n(I-P)^{-1}$, satisfies.

$$a_n^1=\lambda_n^1 \qquad (3.39)$$

$$a_n^i=\lambda_n^i+\sum_{j=1}^{i-1}P_{ji}a_n^j,2\leq i\leq K. \qquad (3.40)$$

(b) : we have

$$W_n^i(k)=(D_n^i)^{-1}(k)-(A_n^i)^{-1}(k),1\leq i\leq i_0. \qquad (3.41)$$

Moreover, by Assumption 4 and Lemma 1, $(B_n(t),\overline{A_n^i}(t),i=1,\cdots,i_0)\xrightarrow{L}$
$(B(t),\overline{A}_i(t),1=1,\cdots,i_0)$ in $D([0,\infty),R^{i_0+v})$
where

$$\overline{A_n^i}(t)=\frac{1}{\sqrt{n}}\left((A_n^i)^{-1}(nt)-\frac{1}{\lambda_n^i}nt\right) \qquad (3.42)$$

$$\overline{A}_i(t)=-\frac{1}{\lambda_i}\hat{A}_i\left(\frac{t}{\lambda_i}\right),1\leq i\leq i_0. \qquad (3.43)$$

Hence, if we set

$$\overline{D_n^i}(t)=\frac{1}{\sqrt{n}}\left((D_n^i)^{-1}([nt])-\frac{1}{\lambda_n^i}nt\right),\quad 1\leq i\leq i_0 \qquad (3.44)$$

$$(B_n(t),\overline{D_n^i}(t),i=1,\cdots,i_0)\xrightarrow{L}(B(t),\hat{W}_i(t)+\overline{A}_i(t),i=1,\cdots,i_0)$$

in $D([0,\infty),R^{i_0+v})$. Then by noting $|D_n^i(t)-(D_n^i)^{-1}(t)|\leq 1$ and by Lemma 1, $(B_n(t),\hat{D}_n^i(t),1=1,\cdots,i_0)\xrightarrow{L}$

$(B(t),\tilde{D}_l(t),l=1,\cdots,i_0)$ in $D([0,\infty),R^{i_0+r})$ where $\tilde{D}_i(t)=-\lambda_i\hat{W}_i(\lambda_i t)-\lambda_i\bar{A}_i(\lambda_i t),1\leq i\leq i_0$. But, $-\lambda_i\bar{A}_i(\lambda_i t)=\hat{A}_i(t)$, and (b) is completed

(2) We show that $(B_n(t),\hat{Q}_n^i(t),l=1,\cdots,i_0)\xrightarrow{L}(B(t),\hat{Q}_l(t),l=1,\cdots,i_0)$ in $D([0,\infty),R^{i_0+r})$. Since, by (3.39),

$$\hat{Q}_n^i(t)=\frac{1}{\sqrt{n}}(A_n^i(nt)-\lambda_n^i nt)-\frac{1}{\sqrt{n}}(D_n^i(nt)-a_n^i nt),$$

$$1\leq i\leq i_0. \tag{3.45}$$

from (1) we have $(B_n(t),\hat{Q}_n^i(t),l=1,\cdots,i_0)\xrightarrow{L}(B(t),\hat{A}_l(t)-\tilde{D}_l(t),l=1,\cdots,i_0)$ in $D([0,\infty),R^{i_0+r})$. But $\hat{A}_i(t)-\tilde{D}_i(t)=\hat{A}_i(t)-(-\lambda_i\hat{W}_i(\lambda_i t)-\lambda_i\bar{A}_i(\lambda_i t))=\lambda_i\hat{W}_i(\lambda_i t)$. Hence we have the conclusion.

Step 2 (1) We prove that if the following two statements (a) and (b) hold for $k=i-1$, then they hold for $k=i$:

(a) for each $t\geq 0, D_n^j(nt)/n\xrightarrow{P}a_j t,\quad 1\leq j\leq k.$

(b) $(B_n(t),\hat{D}_n^j(t),1\leq j\leq k)\xrightarrow{L}(B(t),\tilde{D}_j(t),1\leq j\leq k)$ in $D([0,\infty),R^{K+r})$, where $\tilde{D}_j(t)$ satisfies

$$\tilde{D}_j(t)=-a_jW_j(a_jt)+Z_j(t) \tag{3.46}$$

$$Z_j(t)\equiv\hat{A}_j(t)+\sum_{l=1}^{j-1}(\hat{R}_l(a_lt))_j+\sum_{l=1}^{j-1}P_{lj}\tilde{D}_l(t). \tag{3.47}$$

(a) : By our inductive assumption,

$$\frac{1}{n}(A_n^i(nt)+\sum_{l=1}^{i-1}\sum_{m=1}^{D_n^l(nt)}(R_l(m))_i)\left(=\frac{1}{n}(\alpha_n(nt))_i\right)\xrightarrow{P}$$

$$\lambda_it+\sum_{l=1}^{i-1}a_lP_{li}t=a_it, \tag{3.48}$$

where the last equality is obtained from $(a_1,\cdots,a_K)=\lambda(I-P)^{-1}$ and the assumption $P_{ij}=0\,(i\geq j)$.

Hence $(1/n)(\alpha_n^i)^{-1}(nt)\xrightarrow{P}(1/a_i)t$. Then, $(1/n)W_n^i([nt])\xrightarrow{P}0$, $(1/n)(D_n^i)^{-1}([nt])\xrightarrow{P}(1/a_i)t$ by (3.4)), which in turn implies $(1/n)D_n^i(nt)\xrightarrow{P}a_it$.

(b): Let

$$Z_n^i(t)=\frac{1}{\sqrt{n}}\{A_n^i(nt)+\sum_{l=1}^{i-1}\sum_{m=1}^{D_n^l(nt)}(R_l(m))_i-a_n^int\},$$

$$1\leq i\leq K. \tag{3.49}$$

$$=\frac{1}{\sqrt{n}}(\alpha_n^i(nt)-a_n^int). \tag{3.50}$$

Then, by (3.40)

$$Z_n^i(t)=\frac{1}{\sqrt{n}}(A_n^i(nt)-\lambda_n^int)+\frac{1}{\sqrt{n}}\sum_{l=1}^{i-1}\sum_{m=1}^{D_n^l(nt)}((R_l(m))_i-P_{li})$$

$$+\frac{1}{\sqrt{n}}\sum_{l=1}^{i-1}P_{li}(D_n^l(nt)-a_n^lnt). \tag{3.51}$$

Thus, by Assumption 2 and by our inductive assumptions (a) and (b),

$$(B_n(t),Z_n^j(t),1\leq j\leq i)\xrightarrow{L}(B(t),Z_j(t),1\leq j\leq i) \tag{3.52}$$

in $D([0,\infty),R^{i+r})$ where

$$Z_j(t)\equiv\hat{A}_j(t)+\sum_{l=1}^{j-1}\hat{R}_{lj}(a_lt)-\sum_{l=1}^{j-1}P_{lj}\tilde{D}_l(t),1\leq j\leq i. \tag{3.53}$$

Then by Lemma 1,

$$\left(B_n(t),Z_n^j(t),\frac{1}{\sqrt{n}}\left((\alpha_n^j)^{-1}(nt)-\frac{1}{a_n^j}nt\right),1\leq j\leq i\right)$$

$$\xrightarrow{L}\left(B(t),Z_j(t),-\frac{1}{a_j}Z_j\left(\frac{t}{a_j}\right),1\leq j\leq i\right) \tag{3.54}$$

Therefore by (3.4)

$$\left(B_n(t), Z_n^j(t), \frac{1}{\sqrt{n}}\left((D_n^j)^{-1}([nt]) - \frac{nt}{a_n^j}\right), 1\le j\le i\right)$$
$$\xrightarrow{L} \left(B(t), z_j(t), \hat{W}_j(t) - \frac{1}{a_j} Z_j\left(\frac{t}{a_j}\right), 1\le j\le i\right) \tag{3.55}$$

in $D([0,\infty), R^{i+v})$. Then again by Lemma 1,

$$\left(B_n(t), Z_n^j(t), \frac{1}{\sqrt{n}}(D_n^j(nt) - a_n^j nt), 1\le j\le i\right) \xrightarrow{L}$$
$$(B(t), Z_j(t), \tilde{D}_j, 1\le j\le i) \tag{3.56}$$

in $D([0,\infty), R^{i+v})$, where $\tilde{D}_j(t) = -a_j\hat{W}_j(a_j t) + Z_j(t)$. This complete (b).

(2) We show that if $(B_n(t), \hat{Q}_n^j(t), 1\le j\le i-1) \xrightarrow{L} (B(t), \hat{Q}_j(t), 1\le j\le i-1)$, then $(B_n(t), \hat{Q}_n^j(t), 1\le j\le i) \xrightarrow{L} (B(t), \hat{Q}_j(t), 1\le j\le i)$, where $\hat{Q}_j(t) = a_j\hat{W}_j(a_j t)$. Indeed, note that

$$\hat{Q}_n^i(t) = Z_n^i(t) - \frac{1}{\sqrt{n}}(D_n^i(nt) - a_n^i nt). \tag{3.57}$$

Thus by (3.57)

$$(B_n(t), \hat{Q}_n^j(t), 1\le j\le i) \xrightarrow{L} (B(t), Z_j(t) - \tilde{D}_j(t), 1\le j\le i) \tag{3.58}$$

$$= (B(t), a_j\hat{W}_j(a_j t), 1\le j\le i) \tag{3.59}$$

in $D([0,\infty), R^{i+v})$. □

Combining these theorems, we have the following corollary.

Corollary We assume (1) and (2), and that $P_{ij} = 0$ if $i \ge j$. Then if any one of the following two statments (i) and (ii) hold:

(i) $\left((1/\sqrt{n})Q_n(nt), V_n(t)\right)$ converges weakly, as $n\to\infty$, to a continuous process $(\hat{Q}(t), V(t))$ in $D([0,\infty), R^v)$,

(ii) $((1/\sqrt{n})W_n(nt), V_n(t))$ converges weakly, as $n\to\infty$, to a continuous process $(\hat{W}(t), V(t))$ in $D([0,\infty), R^v)$,

then the other one do too and, in addition, there is the joint convergence

$$\left(\frac{1}{\sqrt{n}}Q_n(nt), \frac{1}{\sqrt{n}}W_n([nt]), V_n(t)\right) \xrightarrow{L} (\hat{Q}(t), \hat{W}(t), V(t)) \tag{3.60}$$

in $D([0,\infty), R^{K+v})$, in which case

$$\hat{W}(t) = \left(\frac{1}{a_1}Q_1\left(\frac{t}{a_1}\right), \cdots, \frac{1}{a_K}Q_K\left(\frac{1}{a_K}\right)\right). \tag{3.61}$$

Remark 2 For the normalization of processes $Q_n(t), W_n(t)$ and others we have used $\sqrt{n}$ (e.g., $(1/\sqrt{n})Q_n(nt)$). However, there is no necessity of using $\sqrt{n}$. Any sequence $\{c_n; n\ge 1\}$ such that $c_n\to\infty$ and $n/c_n\to\infty$ as $n\to\infty$ is sufficient instead of $\sqrt{n}$. We use $\sqrt{n}$ only because in most applications $\sqrt{n}$ is used when limit processes are continuous.

4. Application

We consider the case of Markovian tandem queues where the service rate at each station depends on the queue length of the station. Thus $Q_n(t)$ in (3.1) is modeled as follows:

$$Q_n^1(t) = A_n^1(t) - D_n^1(t) \tag{4.1}$$
$$Q_n^{i(t)} = D_n^{i-1}(t) - D_n^i(t), \quad 2\le i\le K \tag{4.2}$$

(that is, $A_n^i(t)=0$, $2\le i\le K$ and $P_{ij}=1$, if $j=i+1$ and $P_{ij}=0$ if $j\neq i+1$). For that each $n\ge 1$, $A_n^i(t)$, $D_n^i(t), 2\le i\le K$ are counting processes defined on a stochastic basis (Ω_n, F_n, P_n) satisfying usual conditions[3, p2]. They are assumed to have no common discontinuities with probability one, and to have compensators given by

$$\tilde{A}_n^i(t)=\lambda_n^0(t) \tag{4.3}$$

$$\tilde{D}_n^i(t)=\int_0^t \lambda_n^i(Q_n^i(s))ds,\quad 1\le i\le K, \tag{4.4}$$

where $\lambda_n^i(\cdot), 1\le i\le K$, are nonnegative measurable functions with $\lambda_n^i(0)=0$. We make the following assumptions:

Assumption Ⅰ. $\lim_{x\to\infty}\lambda_n^i(x)=\lambda_n^i(\infty)<\infty$ for each n, and $\lim_{\substack{x\to\infty\\ n\to\infty}}\lambda_n^i(x)=\lambda_i<\infty$ for $i=0,1,\cdots,K$ where $\lambda_n^0(x)=\lambda_n^0$. Moreover, $\lambda_n^i(x)\le\lambda_n^{i(\infty)}$ for all $x\ge 0$.

Assumption Ⅱ. $\lim_{\substack{x\to\infty\\ n\to\infty}} x(\lambda_n^i(\infty)-\lambda_n^i(x))=\alpha_i(\ge 0)$, $i=1,\cdots,K$ and $\sup_{n,x}x(\lambda_n^i(\infty)-\lambda_n^i(x))\le M$. Moreover, $\sup_n \lambda_n^i(\infty)<\infty$, for $i=0,1,\cdots,K$

Assumption Ⅲ. $\sqrt{n}(\lambda_n^i(\infty)-\lambda_n^{i-1}(\infty))\to C_i(<\infty)$, $1\le i\le K$, and $\lambda_n^i(\infty)\ge\lambda_n^{i-1}(\infty)$. Hence $c_i\ge 0, 1\le i\le K$

Note that by Assumption Ⅲ, $\lambda_i=\lambda_0$ for $i=1,\cdots,K$. Under Assumptions Ⅰ, Ⅱ and Ⅲ the following fact was proved in [7]: We have

$$A_n(t)\equiv\left(\frac{1}{\sqrt{n}}Q_n(nt),\frac{1}{\sqrt{n}}(A_n^1(nt)-\lambda_n^0(nt))\right) \tag{4.5}$$

$$\xrightarrow{L}(\hat{Q}(t),\hat{A}(t))\, in\, D([0,\infty),R^{c}) \tag{4.6}$$

where $\hat{A}_t=\sqrt{\lambda_0}B_0(t)$ with $B_0(t)$ being a standard Brownian motion and $\hat{Q}(t)=(\hat{Q}_1(t),\cdots,\hat{Q}_K(t))$ is a unique solution of the following Skorohad equation:

$$\hat{Q}_i(t)=-c_it+\sqrt{\lambda_0}B_{i-1}(t)-\sqrt{\lambda_0}B_i(t)+Y_i(t)-Y_{i-1}(t),$$

$$1\le i\le K,\quad Y_0(t)=0 \tag{4.7}$$

where $\hat{Q}(t), 1\le i\le K$ are nonnegative continuous processes, $(B_1(t),\cdots,B_K(t))$ is a K dimensional standard Brownian motion independent of $B_0(t)$ and $Y_i(t), 1\le i\le K$ are nonnegative,nondecreasing processes satisfying $Y_i(0)=0$ and

$$\int_0^t\hat{Q}_i(s)dY_i(s)=\alpha_i t, 1\le i\le K. \tag{4.8}$$

Note that if $\alpha_i=0, i=1,\cdots,K$, (4.8) is equivalent to

$$\int_0^t 1(\hat{Q}_i(s)=0)dY_i(s)=Y_i(t), 1\le i\le K. \tag{4.9}$$

$\hat{Q}(t)$ is an extension of real valued Bessel Processes to a multidimensional case. Let $W_n(t)$ be as Theorem 1 (i.e., $W_n^i(t)=((1/\sqrt{n})W_n^i([nt]),$ $1\le i\le K$) Then in view of Theorem 1, we have

$$\left(\frac{1}{\sqrt{n}}(A_n^1(nt)-\lambda_n^0 nt), W_n(t)\right)\xrightarrow{L}(\hat{A}(t),\hat{W}(t)) \tag{4.10}$$

in $D([0,\infty),R^{1+c})$ where $\hat{W}(t)$ is given by $\hat{W}(t)=((1/\lambda_0)\hat{Q}_1((t/\lambda_0)),\cdots,(1/\lambda_0)\hat{Q}_K((t/\lambda_0) right))$.
Moreover, by Theorem 2, (4.5) is necessary and sufficient for (4.10) to hold. $\hat{W}(t)$ satisfies the following Skorohod equation analogous to

(4.7):

$$(t) = -c_i \frac{1}{\lambda_0^2} t + \frac{1}{\sqrt{\lambda_0}} B_{i-1} \left(\frac{t}{}\lambda_0\right) - \frac{1}{\sqrt{\lambda_0}} B_i \left(\frac{t}{\lambda_0}\right) +$$

$$\overline{Y_i}(t) - \overline{Y_{i-1}}(t), \quad 1 \leq i \leq K \qquad (4.11)$$

where $\overline{Y_i}(t), 1 \leq i \leq K$ satisfy $\overline{Y}(0) = 0$ and

$$\int_0^t \hat{W}_i(s) d\overline{Y_i}(s) = \alpha_i \frac{1}{\lambda_0^2} t. \qquad (4.12)$$

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5. Conclusions

We investigated, in a setting general as much as possible, the relationship between weak limits of the queue length and waiting time at stations of queueing systems under heavy traffic situations. It was shown that under suitable normalizaion, weak convergence of queue lengths and arrival processes is a sufficient condition for that of waiting times, and is also necessary condition when the network is of feedforward type. Moreover, these weak limits for queue lengths and waiting times were shown to be simply related.

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