▩ 연구보고

k-Sample Rank Procedures for Ordered Location-Scale Alternatives

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Abstract

Some rank score tests are proposed for testing the equality of all sampling distribution functions against ordered location-scale alternatives in k-sample problem. Under the null hypothesis and a contiguous sequence of ordered location-scale alternatives, the asymptotic properties of the proposed test statistics are investigated. Also, the asymptotic local powers are compared with each others. The results show that the proposed tests based on the Hettmansperger-Norton type statistic are more powerful than others for the general ordered location-scale alternatives. However, the Shiraishi's tests based on the sum of two Bartholomew's rank analogue statistics are robust.

1. Introduction

Let $X_{i1}, \cdots, X_{in}$ be k independent random samples from continuous distribution functions $F_i(x)$, $i=1, \cdots, k$, respectively. In this k-sample model, we want to propose test procedures for testing the null htpothesis

$$H_0 : F_1(x) = \cdots = F_k(x), \quad \text{for all } x. \tag{1.1}$$

Kruskal and Wallis (1952) proposed a distribution-free test based on Wilcoxon (1945) rank statistic for the general alternative. For the ordered alternatives, which is of the form

$$H_1 : F_1(x) \geq \cdots \geq F_k(x), \quad \text{for all } x, \tag{1.2}$$

Jonckheere (1954) and Terpstra (1952) independently proposed a distribution-free test based on the pairwise Mann-Whitney (1947)-Wilcoxon (1945) statistics. Chacko (1963) suggested a rank analogue of the Bartholomew's (1959) likelihood ratio test. Shirahata (1980) proposed a maximin efficient linear rank procedures. Hettmansperger and Norton (1987) proposed a linear rank test having the maximum local power and the greatest efficiency, when the pattern is specified. In these articles, they proposed rank procedures against only differences in locations not including scale parameters. But in usual k-sample problem, the experimenter often encounters with the cases that the variance increases as the mean increases. Hence the experimenter would do better to consider rank test which reacts well to the differences in both location and scale parameters. So, we let $F_i(x)=F((x-\theta_i)/\sigma_i)$, where θ_i and σ_i denote unknown location and scale parameter of ith population, $i=1, \cdots, k$, respectively, and consider the ordered location-scale alternative, which is of the form

$$H_a : \theta_1 \leq \cdots \leq \theta_k \text{ and } \sigma_1 \leq \cdots \leq \sigma_k, \tag{1.3}$$

with at least one strict inequality. Shiraishi (1988) proposed some rank tests for the ordered location-scale alternatives. We want to seek rank procedures which have good peformances on the asymptotic (local) power and compare with the rank tests in Shiraishi (1988).

In Section 2, some rank tests based on (random vectors consisting of) the simple linear rank score statistics are proposed, where each of the linear rank score statistics is sensitive to the location or scale alternatives. Also, the asymptotic properties of the proposed test statistics are investigated under the null hypothesis and the contiguous sequence of ordered location-scale alternatives. In Section 3 an example is suggested and the proposed tests are compared with the rank tests in Shiraishi (1988) by the asymptotic (or empirically estimated) local powers. The results show that the tests based on the Hettmansperger-Norton (1987) type statistic are more powerful than others for the general ordered location-scale alternatives. However, the Shiraishi's tests based on the sum of two Bartholomew's (1959) rank analogue statistics are highly stable for all the investigated cases.

2. Proposed tests and their asymptotic properties

Let $X_{i1}, \cdots, X_{in}$ be k independent random samples of the same size from continuous distribution functions $F_i(x)=F((x-\theta_i)/\sigma_i)$, where θ_i and σ_i denote unknown location and scale parameter of ith population, $i=1, \cdots, k$, respectively. We want to test the null hypothesis H_0 against the ordered location-scale alternatives H_a. In order to propose some rank test statistics having good performances, we would do better to use rank tests sensitive to the differences in both location and scale parameters. Thus we define ranks and define two rank score functions which are sensitive to the differences for the ordered location-scale alternatives. Let R_{ij} be the rank of X_{ij}, $i=1, \cdots, k$, $j=1, \cdots, n$, among the overall combined observations X_{ij}'s. Setting $N=kn$, let $a_N(\cdot)$ and $b_N(\cdot)$ be real valued functions defined on $\{1, \cdots, N\}$ satisfying

$$\lim_{N\to\infty}\int_0^1 \{a_N(1+[uN])-\phi(u)\}^2 du=0, \quad 0<u<1 \tag{2.1}$$

and

$$\lim_{N\to\infty}\int_0^1 \{b_N(1+[uN])-\psi(u)\}^2 du=0, \quad 0<u<1 \tag{2.2}$$

with $[x]$ denoting the largest integer not exceeding x, for some square integrable non-constant score generating functions ϕ and ψ. (2.1) implies $a_N(i)=\phi(i/(N+1))$, $i=1, \cdots, N$ and similarly (2.2) does. Furthmore, we assume, for $m=1, \cdots, N$,

$$a_N(N-m+1)=-a_N(m), \quad b_N(N-m+1)=b_N(m) \tag{2.3}$$

and

$$a_N(1)\le a_N(2)\le\cdots\le a_N([N/2])\le 0, \ b_N(1)\ge b_N(2)\ge\cdots\ge b_N([N/2]). \tag{2.4}$$

Score $a_N(\cdot)$ in (2.3) shows $\sum_{m=1}^{N} a_N(m)=0$. Also, assume $\sum_{m=1}^{N} b_N(m)=0$ for simplicity of the score for scale alternatives. Then the equations (2.1) through (2.4) imply

$$\phi(1-u)=-\phi(u) \quad \text{and} \quad \psi(1-u)=\psi(u), \quad \text{for } u\in(0, 1)$$

which give

$$\int_0^1\phi(u)du=0 \quad \text{and} \quad \int_0^1\phi(u)\psi(u)du=0, \quad \text{for } u\in(0, 1).$$

These two scores $a_N(\cdot)$ and $b_N(\cdot)$ are often used in rank tests for the location and scale alternatives. For example, $R_{ij}/(N+1)-1/2$ (Wilcoxon) or $\Phi^{-1}(R_{ij}/(N+1))$ (Normal), where Φ is the cdf of standard normal, is one for score function $a_N(R_{ij})$, and $|R_{ij}/(N+1)-1/2|-N/4(N+1)$ (Ansari-Bradley) or $(R_{ij}/(N+1)-1/2)^2-(N-1)/12(N+1)$ (Mood) is one for $b_N(R_{ij})$. For the details of scores $a_N(\cdot)$ and $b_N(\cdot)$, see Randles and Wolfe (Chapter 9, 1979). Let

$$\mathbf{S} = (S_1, \cdots, S_k)' \quad \text{and} \quad \mathbf{T} = (T_1, \cdots, T_k)'$$

with

$$S_i = \frac{\sum_{j=1}^{n} a_N(R_{ij})}{\sqrt{n \sum_{m=1}^{N} a_N^2(m) / (N-1)}} \quad \text{and} \quad T_i = \frac{\sum_{j=1}^{n} b_N(R_{ij})}{\sqrt{n \sum_{m=1}^{N} b_N^2(m) / (N-1)}},$$

for $i=1, \cdots, k$. Then S_i and T_i are the simple linear rank score statistic sensitive to the difference in location and scale parameters, $\bar{S} = \sum_{i=1}^{k} S_i/k = 0$ and $\bar{T} = \sum_{i=1}^{k} T_i/k = 0$. Also, under the null hypothesis H_0, $E_0(\mathbf{S}) = E_0(\mathbf{T}) = 0$, $Var_0(\mathbf{S}) = Var_0(\mathbf{T}) = \Sigma$ and $Cov(S_i, T_j) = 0$ for all i's and j's, where $E_0(\cdot)$, $Var_0(\cdot)$ and $Cov_0(\cdot, \cdot)$ denote the expectation, the variance-covariance matrix and the covariance, respectively, and $\Sigma_k = \mathrm{I}_k - \mathbf{1}\cdot\mathbf{1}'/k$, where I_k is the unit matrix of order k, and $\mathbf{1} = (1, \cdots, 1)'$.

The proposed test statistics based on (the random vectors consisting of) the simple linear rank score statistics S_i or T_i, $i=1, \cdots, k$, are stated as follows. In k-sample location problem, Hettmansperger and Norton (1987) proposed an optimal linear rank test $\sum_{i=1}^{k}(c_i-\bar{c})S_i/\{\sum_{i=1}^{k}(c_i-\bar{c})^2\}^{1/2}$ having the maximum local power and the greatest efficiency, when the patterns $\{c_i\}$ are specified. But generally the patterns are unspecified, so Hettmansperger and Norton recommended the weights $c_i = i$, for $i=1, \cdots, k$ and $\bar{c} = \sum_{i=1}^{k} c_i/k$ for ordered location alternatives having the same scale parameters. Thus they have

$$HN = \frac{\sum_{i=1}^{k}\{i-(k+1)/2\}S_i}{\sqrt{\sum_{i=1}^{k}\{i-(k+1)/2\}^2}}.$$

For ordered location-scale alternatives, the orderings of both location and scale parameters should be considered. For ordered scale alternatives having the same location parameters, $T_1 \leq \cdots \leq T_k$ is expected as $S_1 \leq \cdots \leq S_k$ is expected for ordered location alternatives having the same scale parameters. Hence, the proposed test statistics are the Hettmansperger-Norton type statistics defined by

$$HN^* = \frac{\sum_{i=1}^{k} \{i-(k+1)/2\}(S_i + T_i)}{\sqrt{2}\sqrt{\sum_{i=1}^{k} \{i-(k+1)/2\}^2}}, \qquad (2.5)$$

which is based on the sum of two HN for S_i and T_i. H_0 is rejected for large values of HN^*.

Under the null hypothesis H_0, HN^* has asymptotically standard normal distribution. In order to investigate the asymptotic properties under local alternatives, consider the sequence of ordered location-scale alternatives

$$H_{aN} : F_i(x) = F\left(\frac{x-\delta_i/\sqrt{n}}{e^{\omega_i/\sqrt{n}}} \right), \qquad (2.6)$$

where

$$\delta_1 \leq \cdots \leq \delta_k \quad \text{and} \quad \omega_1 \leq \cdots \leq \omega_k, \qquad (2.7)$$

with at least one strict inequality. Let $p_n(x)$ and $q_n(x)$ are the joint density functions of observations X_{ij}'s under H_0 and H_{aN}, respectively, then

$$p_n(x) = \prod_{i=1}^{k} \prod_{j=1}^{n} f(x_{ij}),$$

$$q_n(x) = \prod_{i=1}^{k} \prod_{j=1}^{n} \frac{1}{e^{\omega_i/\sqrt{n}}} f\left(\frac{x_{ij}-\delta_i/\sqrt{n}}{e^{\omega_i/\sqrt{n}}} \right),$$

where $f(x) = F'(x)$ and assume that $f(x)$ is symmetric about 0. Setting $d_i(x, \theta) = f((x-\delta_i\theta)/e^{\omega_i\theta})/e^{\omega_i\theta}$, and impose the following Assumption for sufficient condition for contiguity (Hájek and Sidák, 1967) in Shiraishi (1988).

Assumption

$$\lim_{\theta\to 0}\int_{-\infty}^{\infty}\left\{\frac{\sqrt{d_i(x,\theta)}-\sqrt{d_i(x,0)}}{\theta}-\frac{d_i'(x,0)}{2\sqrt{d_i(x,0)}}\right\}^2 dx = 0,$$

where $d_i'(x,0) = -\omega_i f(x) - (\delta_i + \omega_i x) f'(x)$.

With the Assumption, the family of densities $\{q_n(x)\}$ is contiguous to that of densities $\{p_n(x)\}$ as $N\to\infty$, from Shiraishi (1988). Moreover, the following theorem is obtained.

Theorem (Shiraishi) Suppose that Assumption is satisfied. Then, under H_{aN} and as $n\to\infty$, $(\mathbf{S}', \mathbf{T}')'$ has asymptotically a multivariate normal distribution with mean $(\nu', \xi')'$ and variance-covariance matrix Γ, where

$$\nu = (\nu_1, \cdots, \nu_k)',\ \xi = (\xi_1, \cdots, \xi_k)',$$

$$\nu_i = -(\delta_i - \bar{\delta})\int_{-\infty}^{\infty} f'(x)\phi(F(x))dx/C(\phi), \tag{2.8}$$

$$\xi_i = -(\omega_i - \bar{\omega})\int_{-\infty}^{\infty} xf'(x)\psi(F(x))dx/C(\psi), \tag{2.9}$$

$$C(\phi) = \{\int_0^1 \phi^2(u)du\}^{1/2},\ C(\psi) = \{\int_0^1 \psi^2(u)du\}^{1/2},$$

$$\Gamma = \mathrm{I}_2 \otimes \Sigma_k,$$

where $A\otimes B$ denotes the Kronecker products of A and B, $\bar{\delta} = \sum_{i=1}^{k}\delta_i/k$ and $\bar{\omega} = \sum_{i=1}^{k}\omega_i/k$.

From $\int_{-\infty}^{\infty} f'(x)\phi(F(x))dx<0$, $\int_{-\infty}^{\infty} xf'(x)\psi(F(x))dx<0$, (2.8) and (2.9), we know that the ordered location-scale alternatives H_{aN} defined by (2.6) and (2.7) rewrite to the modified ordered location-scale alternatives, which is of the form

$$H_a^* : \nu_1 \le \cdots \le \nu_k, \ \text{and}\ \ \xi_1 \le \cdots \le \xi_k, \tag{2.10}$$

with at least one strict inequality and $\sum_{i=1}^{k}\nu_i = \sum_{i=1}^{k}\xi_i = 0$. Now regard $(\mathbf{S}', \mathbf{T}')'$ as the random variable having multivariate normal distribution with mean$(\nu', \xi')'$ and variance-covariance matrix $\Gamma = \mathrm{I}_2 \otimes \Sigma_k$ and try to seek highly powerful tests for the modified null hypothesis $H_0^* : \nu_i = \xi_i = 0$, $i = 1, \cdots, k$, against H_a^*.

From Theorem, under $H_a^{\cdot}$ and Assumption,

$$\lim_{n\to\infty} Pr(HN^* \geq t) = 1-\Phi(t-\mu_{HN^*}), \qquad (2.11)$$

where

$$\mu_{HN^*} = \frac{\sum_{i=1}^{k}\{i-(k+1)/2\}(\nu_i+\xi_i)}{\sqrt{2}\sqrt{\sum_{i=1}^{k}\{i-(k+1/2)\}^2}}$$

is the expectation of test HN^* under $H_a^{\cdot}$.

3. Asymptotic power comparison

Now compare the proposed test statistic with the rank tests in Shiraishi (1988), given by the followings

$$ST = \sum_{i=1}^{k}(\hat{S}_i^2 + \hat{T}_i^2), \qquad (3.1)$$

$$\bar{\chi}^2_{k,a(k)} = \sum_{i=1}^{k}\hat{S}_i^2, \qquad (3.2)$$

$$SH^* = \frac{(S_k - S_1 + T_k - T_1)}{2}, \qquad (3.3)$$

$$SH = \frac{(S_k - S_1)}{\sqrt{2}}, \qquad (3.4)$$

$$LN = \sum_{i=1}^{k}(S_i^2 + T_i^2), \qquad (3.5)$$

where $\hat{S}_1 \leq \cdots \leq \hat{S}_k$ and $\hat{T}_1 \leq \cdots \leq \hat{T}_k$ are the isotonic regression estimators of $S_1, \cdots, S_k$ and $T_1, \cdots, T_k$ with $\sum_{i=1}^{k}\hat{S}_i/k=0$ and $\sum_{i=1}^{k}\hat{T}_i/k=0$. The algorithm to derive the isotonic regression estimators is discussed in Barlow et al. (1972).

An example is suggested to compute the value of the proposed and competing test statistics. The rank data derived from four samples are $\mathbf{R}_1=(4, 5, 6, 9, 12)'$, $\mathbf{R}_2=(3, 7, 10, 13, 16)'$, $\mathbf{R}_3=(2, 8, 14, 15, 18)'$ and $\mathbf{R}_4=(1, 11, 17, 19, 20)'$, where $\mathbf{R}_i=(R_{i1}, R_{i2}, \cdots, R_{i5})'$, $i=1, \cdots, 4$, respectively. Using $a_N(i)=i/(N+1)-1/2$ and $b_N(i)=|i/(N+1)-1/2|-N/4(N+1)$, we get $\sum_{m=1}^{N} a_N^2(m)=N(N-1)/12(N+1)$ and $\sum_{m=1}^{N} b_N^2(m)=N(N+2)(N-2)/48(N+1)^2$. So, we have $10\sqrt{7}$ $\mathbf{S}=(-33, -7$

9, 31)′, $10\sqrt{33/19}$ $\mathbf{T}=(-11, -11, 3, 19)'$ and $ST=6.638$, $\bar{\chi}^2_{k,\alpha(R)}=3.114$, $SH^*=2.348$, $SH=1{,}710$, $LN=6.638$, $HN^*=3.409$ and $HN=1.758$ are obtained.

The asymptotic properties and characteristics of the test statistics mentioned above are stated in Shiraishi (1988). we calculate the asymptotic powers of tests and compare with other tests. First, fix the common significance level α and the asymptotic power β of the test based on LN, which is optimal for the general location-scale alternatives. Second, consider the asymptotic power function of the test based on LN and choose some values of the pair (ν, ξ) satisfying $\eta_L^2=\sum_{i=1}^{k}(\nu_i^2+\xi_i^2)$ with $\nu_i=\nu_i(\delta_i, \phi, f)$, $\xi_i=\xi_i(\omega_i, \psi, f)$ (given in (2.8), (2.9)) and

$$Pr\{\chi^2_{2k-2}(\eta_L^2)\geq \chi^2_{2k-2,\alpha}\}=\beta,$$

where $\chi^2_{2k-2,\alpha}$ is the upper 100α percentile of the central chi-square distribution with $2k-2$ degrees of freedom. Finally, applying the chosen value (ν, ξ) into the asymptotic power functions of the test statistics HN, HN^* and (3.1)~(3.4), we get the asymptotic powers and compare them. Note that the asymptotic powers of the tests based on ST, HN^*, SH^* and LN are invariant to the changes of chosen values ν_i and ξ_i, $i=1, \cdots, k$. But the asymptotic powers of the tests based on $\bar{\chi}^2_{k,\alpha(R)}$, HN and SH are not, and does not depend on the difference of scale parameters ξ_i, $i=1, \cdots, k$. In other words, the tests based on $\bar{\chi}^2_{k,\alpha(R)}$, HN and SH are tests for the ordered location alternatives without including scale alternatives. Thus letting $\gamma=\xi_i/\nu_i$ for $i=1, \cdots, k$, the asymptotic powers of the test statistics based on ST, $\bar{\chi}^2_{k,\alpha(R)}$, HN^*, HN, SH^* and SH are given in Table 1 for $\alpha=0.05$; $\beta=0.5$; $\gamma=0.0$, 0.5, 1.0; k-sorts of specified ordered location-scale alternatives; $k=3$, 4, 5. Since the asymptotic powers of test based on $\bar{\chi}^2_{k,\alpha(R)}$ and ST are complicated to calculate, they are empirically estimated by a small-sample Monte-Carlo simulation from 1,000 samples which are generated by the IMSL subroutine RNMVN.

Table 1 shows that the asymptotic (or empirically estimated) powers of the tests based on ST, HN^* and SH^* are larger than those of $\bar{\chi}^2_{k,\alpha(R)}$, HN and SH, respectively, except $\gamma=0.0$, which means the ordered location alternatives having the same scale parameters. Also we know that the proposed tests based on HN^* are more powerful than others for the general investigated ordered location-scale alternatives (that is, $\gamma\neq 0.0$). However, the tests based on ST are highly stable to the change of scale parameters for all the investigated cases. Thus, we can conclude that the tests based on ST are robust.

Table 1. Asymptotic Powers of *ST*, $\bar{\chi}^2_{k,\alpha(R)}$, *HN** *SH**, *HN* and *SH* of Level α=0.05 Relative to Asymptotic Power β=0.5 of *LN*

k	ν	γ	ST	$\bar{\chi}^2_{k,\alpha(R)}$	HN^*	SH^*	HN	SH
			alternative: $\nu' = (-2\nu, \nu, \nu)$					
3	1.034	0.0	0.632	0.731	0.463	0.463	0.707	0.707
	0.766	0.5	0.546	0.520	0.532	0.532	0.492	0.492
	0.731	1.0	0.684	0.495	0.707	0.707	0.463	0.463
			alternative: $\nu' = (-\nu, -\nu, 2\nu)$					
	1.034	0.0	0.635	0.727	0.463	0.463	0.707	0.707
	0.766	0.5	0.578	0.555	0.532	0.532	0.492	0.492
	0.731	1.0	0.681	0.478	0.707	0.707	0.463	0.463
			alternaive: $\nu' = (-\nu, 0.0, \nu)$					
	1.790	0.0	0.720	0.803	0.557	0.557	0.812	0.812
	1.600	0.5	0.726	0.680	0.775	0.775	0.754	0.754
.	1.267	1.0	0.747	0.526	0.813	0.813	0.559	0.559
			alternative: $\nu' = (-\nu, 0.0, 0.0, \nu)$					
4	1.940	0.0	0.724	0.815	0.577	0.578	0.831	0.864
	1.730	0.5	0.779	0.744	0.792	0.829	0.751	0.788
	1.370	1.0	0.787	0.562	0.830	0.863	0.578	0.579
			alternative: $\nu' = (-4\nu, -2\nu, 2\nu, 4\nu)$					
	0.443	0.0	0.721	0.817	0.608	0.534	0.856	0.790
	0.387	0.5	0.787	0.766	0.822	0.750	0.781	0.707
	0.306	1.0	0.795	0.549	0.857	0.789	0.606	0.534
			alternaive: $\nu' = (-2\nu, -2\nu, 0.0, 4\nu)$					
	0.559	0.0	0.702	0.795	0.549	0.513	0.804	0.766
	0.500	0.5	0.758	0.716	0.766	0.727	0.722	0.647
	0.395	1.0	0.762	0.524	0.803	0.766	0.548	0.513
			alternaive: $\nu' = (-3\nu, -\nu, \nu, 3\nu)$					
	0.613	0.0	0.745	0.836	0.615	0.577	0.863	0.830
	0.548	0.5	0.799	0.741	0.830	0.794	0.790	0.752
	0.433	1.0	0.799	0.566	0.863	0.830	0.614	0.576

Table 1. (continued)

k	ν	γ	ST	$\bar{\chi}^2_{k,a(R)}$	HN^*	SH^*	HN	SH
			alternative: $\nu' = (-4\nu, -4\nu, \nu, \nu, 6\nu)$					
5	0.350	0.0	0.746	0.852	0.623	0.542	0.869	0.797
	0.310	0.5	0.804	0.768	0.830	0.719	0.790	0.707
	0.250	1.0	0.825	0.584	0.876	0.803	0.630	0.548
			alternative: $\nu' = (-2\nu, -2\nu, 0.0, \nu, 3\nu)$					
	0.680	0.0	0.770	0.842	0.630	0.522	0.875	0.776
	0.610	0.5	0.817	0.774	0.845	0.740	0.806	0.696
	0.480	1.0	0.807	0.579	0.874	0.775	0.630	0.522
			alternaive: $\nu' = (-8\nu, -3\nu, 2\nu, 2\nu, 7\nu)$					
	0.255	0.0	0.758	0.844	0.637	0.605	0.880	0.855
	0.225	0.5	0.809	0.761	0.840	0.812	0.801	0.770
	0.180	1.0	0.832	0.617	0.879	0.854	0.635	0.605
			alternaive: $\nu' = (-3\nu, -\nu, 0.0, \nu, 3\nu)$					
	0.650	0.0	0.790	0.870	0.652	0.619	0.891	0.865
	0.580	0.5	0.818	0.771	0.859	0.833	0.822	0.792
	0.460	1.0	0.852	0.623	0.891	0.866	0.652	0.618
			alternaive: $\nu' = (-2\nu, -\nu, 0.0, \nu, 2\nu)$					
	0.920	0.0	0.780	0.871	0.659	0.576	0.897	0.830
	0.820	0.5	0.811	0.784	0.865	0.792	0.829	0.750
	0.650	1.0	0.849	0.625	0.896	0.830	0.659	0.576

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