Optimization of a Model for an Inventory with Poisson Restocking[+]

−Optimization of an Inventory Model−

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Abstract

An inventory supplies stock continuously at a constant rate. A deliveryman arrives according to a Poisson process. If the level of the inventory, when he arrives, exceeds a threshold, no action is taken, otherwise a delivery is made by a random amount. Costs are assigned to each visit of the deliveryman, to each delivery, to the inventory being empty and to the stock being kept. It is shown that there exists a unique arrival rate of the deliveryman which minimizes the average cost per unit time over an infinite horizon.

1. Introduction

Baxter and Lee (1987) introduced a model for an inventory with constant demand and Poisson restocking. The inventory supplies stock continuously at constant rate $\mu > 0$, and, if the inventory becomes exhausted, its state remains at 0 until a restocking occurs; this is done by means of a deliveryman who arrives according to a Poisson process of rate $\lambda > 0$. If the level of the inventory, when he arrives, exceeds a threshold $\alpha > 0$, no stock is delivered, otherwise, a delivery is made. It is assumed that the available size of a delivery is a random variable Y ($Y \geq \alpha$ almost surely). Baxter and Lee (1987) gave an application of the model and studied the distribution of $X(t)$, the level of the inventory at time t, obtaining

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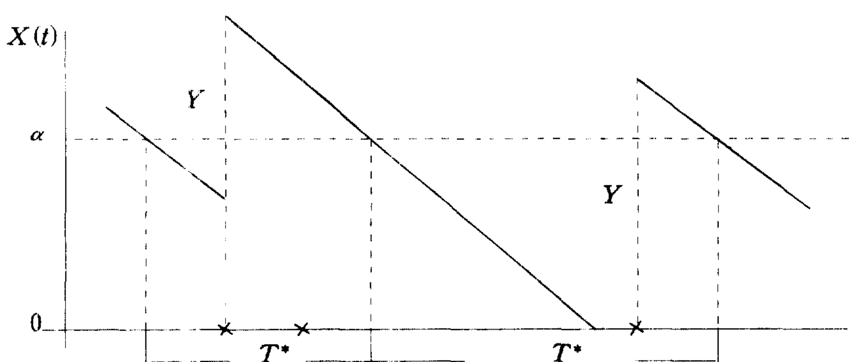
explicit results for the stationary case.

In this paper, we extend the earlier analysis by assigning costs to each visit of the deliveryman, to each delivery, to the inventory being empty, and to the stock being kept, and then seeking to minimize the average cost by varying arrival rate λ. Optimization of the inventory models has been studied by many authors. See Silver and Peterson (1985) and the references thereon. Most works, however, have been concentrated on the optimal policy with respect to the threshold. In the present paper, we show that there exists a unique arrival rate λ which minimizes the average cost per unit time over an infinite horizon.

2. Optimization

It is assumed throughout that the process $\{ X(t), t \geq 0 \}$ is stationary. Let C_1 denote the cost per visit of the deliveryman, let C_2 denote the cost of ordering a unit of the stock, let C_3 denote the cost per unit time of the inventory being empty, and let C_4 denote the cost per unit time of keeping a unit of the stock. We calculate $C(\lambda)$, the average cost per unit time over an infinite horizon for a given arrival rate λ. To do so, we define a cycle T^*, illustrated in Figure 1, as the interval between two successive instants at which the level of the inventory crosses the threshold α.



× : a Poisson process of the deliveryman

Figure 1

Baxter and Lee (1987) observed that the sequence of such instants forms an embeded renewal process with

$$E(T^*) = m_1/\mu + e^{-\alpha\lambda/\mu}/\lambda,$$

where $m_1 = E(Y)$.

We, first, consider the costs in a cycle. Note that the number of visits of the deliveryman during a cycle is a Poisson random variable with mean $\lambda E(T^*)$ and the amount of a delivery is the random variable Y. Let T denote the duration of the inventory being in state 0 during a cycle, then it can be shown that

$$E(T) = e^{-\alpha\lambda/\mu}/\lambda,$$

due to the memoryless property of the exponential distribution. To calculate the keeping cost of the stock, define S as the total time of the whole stock being kept during a cycle. Then it will be the area under the process $X(t)$ in a cycle (See Figure 1) and hence is given by

$$S = Y^2/2\mu + X'Y/\mu,$$

where X' is the level of the stock immediately prior to a delivery. Baxter and Lee (1987) showed that

$$E(X') = \alpha + \mu(e^{-\alpha\lambda/\mu} - 1)/\lambda.$$

Let $\tilde{C}(\lambda) = C_1\lambda E(T^*) + C_2 m_1 + C_3 e^{-\alpha\lambda/\mu}/\lambda$
$+ C_4[m_2/2\mu + m_1\{\alpha + \mu(e^{-\alpha\lambda/\mu} - 1)/\lambda\}]$,

where $m_2 = E(Y^2)$. Then $C(\lambda) = \tilde{C}(\lambda)/E(T^*)$ and it follows that

$$C(\lambda) = C_1\lambda + \frac{C_2 m_1\mu\lambda + C_3\mu e^{-\alpha\lambda/\mu} + C_4\,[m_2\lambda/2 + m_1\{\alpha\lambda + \mu(e^{-\alpha\lambda/\mu} - 1)\}]}{m_1\lambda + \mu e^{-\alpha\lambda/\mu}}$$

and its derivative is given by

$$C'(\lambda) = \frac{A(\lambda) - B(\lambda)}{(m_1\lambda + \mu e^{-\alpha\lambda/\mu})^2},$$

where

$$A(\lambda) = C_1(m_1\lambda + \mu e^{-\alpha\lambda/\mu})^2 + C_4 m_1(m_1\mu - m_1\mu e^{-\alpha\lambda/\mu} + \alpha^2\lambda e^{-\alpha\lambda/\mu} - \alpha m_1\lambda e^{-\alpha\lambda/\mu})$$

and

$$B(\lambda) = (-C_2 m_1 \mu + C_3 m_1 - C_4 m_2/2)(\mu e^{-\alpha\lambda/\mu} + \alpha\lambda e^{-\alpha\lambda/\mu}).$$

Theorem 1 If $C_1\mu + C_2 m_1 \mu + C_4 m_2/2 \geq C_3 m_1$, then $C(\lambda)$ achives its minimum value C_3 at $\lambda = 0$, otherwise there exists a unique λ^* $(0 < \lambda^* < \infty)$ which minimizes $C(\lambda)$.

Proof. Suppose, firstly, that $C_1\mu + C_2 m_1 \mu + C_4 m_2/2 \geq C_3 m_1$. Then

$$\begin{aligned} C(\lambda) - C(0) &\geq C_1\lambda + \frac{C_2 m_1 \mu\lambda + C_3 \mu e^{-\alpha\lambda/\mu} + C_4 m_2 \lambda}{m_1\lambda + \mu e^{-\alpha\lambda/\mu}} \; 2 - C_3, \\ &\text{since } e^x - 1 \geq x \ \forall x \in R, \\ &\geq \frac{C_1\lambda(m_1\lambda + \mu e^{-\alpha\lambda/\mu} - \mu)}{m_1\lambda + \mu e^{-\alpha\lambda/\mu}}, \\ &\geq 0, \text{ since } m_1 \geq \alpha. \end{aligned}$$

Suppose, now, that $C_1\mu + C_2 m_1 \mu + C_4 m_2/2 < C_3 m_1$. Then

$$\lim_{\lambda\to 0} A(\lambda) = C_1\mu < -C_2 m_1 \mu + C_3 m_1 - C_4 m_2/2 = \lim_{\lambda\to 0} B(\lambda),$$

and $\lim_{\lambda\to\infty} A(\lambda) = \infty$ and $\lim_{\lambda\to\infty} B(\lambda) = 0$. Further, a routine calculation shows that $A'(\lambda) > 0$ and $B'(\lambda) < 0$. Therefore, there exists a unique λ^* such that $A'(\lambda^*) = B(\lambda^*)$, which minimizes $C(\lambda)$.

Remarks. (i) Notice that the left side of the condition in the above theorem is related to the maintenance cost of the inventory such as the ordering and keeping costs of the stock, meanwhile, the right side is related to the penalty of the inventory being empty.

(ii) When $\alpha = 0$, under the condition that $C_1\mu + C_2 m_1 \mu + C_4 m_2/2 < C_3 m_1$, λ^* can be explicitly calculated as

$$\lambda^* = \frac{-C_1\mu + \sqrt{-C_1\mu(C_2 m_1 \mu - C_3 m_1 + C_4 m_2/2)}}{C_1 m_1}$$

References

[1] Baxter, L. A. and Lee, E. Y.(1987), An Inventory with Constant Demand and Poisson Restocking. *Probability in the Engineering and Informational Sciences*, Vol. 1, 203－210.

[2] Silver, E. A. and Peterson, R.(1985), *Decision Systems for Inventory Management and Production Planning*, John Wiley and Sons, New York.