Asymptotic Inferences on the Shape Parameter of a Gamma Distribution : An Unconditional Approach

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Abstract

In this paper we develop an unconditional method for inferences on the shape parameter of a gamma distribution. A simple numerical implementation of this unconditional method is developed; this is a computer program that takes the observed data as input and produces the confidence distribution function for the shape parameter, which in turn provides approximate observe significance levels and confidence intervals for that parameter, as output. These approximations are extremely accurate even for very small sample size and numerically simple and easy to obtain.

1. Introduction

Consider the two-parameter gamma distribution with shape parameter α and scale parameter β, i.e., with the density

$$f(t;\alpha,\beta)=\{\Gamma(\alpha)\beta^{\alpha}\}^{-1}\cdot\exp\{\alpha\log t-t/\beta\}/t,\qquad t>0. \tag{1.1}$$

The shape parameter α is especially of interest to reliability analysts because the gamma distribution has decreasing failure rate (DFR), constant, or increasing failure rate (IFR) according to whether $(\alpha-1)$ is negative, zero, or positive. Unfortunately, the shape parameter is generally unknown when distributions are fitted to real data.

For inferences about α, many methods have been proposed and were reviewed by Keating, Glaser and Ketchum (1990). But, they used the Cray computer to obtain extensive tables of critical values for various sample sizes and values of α. Recently, Wong (1992) proposed a new conditional method based on saddlepoint approximation due to Barndorff-Nielsen (1990). This conditional inference about α is simple, accurate, and easy to use for converting the observed data to observed significance level and confidence interval for any chosen level of interest. In this paper, we suggest a new unconditional method for inference on α and compare it with others through examples.

2. Inference on the Shape Parameter

Suppose $T_1, \cdots, T_n$ constitute random samples of size n from a density (1.1). Let $\bar{T} = n^{-1}\sum_{i=1}^{n} T_i$ is sample arithmetic mean and $T = (\prod_{i=1}^{n} T_i)^{1/n}$ is the sample geometric mean. Define a new statistic

$$W = \log(T/T). \qquad (2.1)$$

Then $(W, \bar{T})$ is a minimal sufficient statistic for (α, β) with W and $\bar{T}$ being stochastically independent. (see Glaser (1976)). The cumulant generating function (CGF) of W is given by

$$\begin{aligned} K_W(\xi) = & -\xi \log n + n \log \Gamma(\alpha - \xi/n) - \log \Gamma(n\alpha - \xi) \\ & - n \log \Gamma(\alpha) + \log \Gamma(n\alpha), \end{aligned} \qquad (2.2)$$

where $\Gamma(\cdot)$ is gamma function. Note that (2.2) depends only on the shape parameter α. In this paper, we will consider the CGF of W only for inference about α. As we know the explicit form of CGF $K_W(\xi)$, Lugannani and Rice's (1980) results can be applied to approximate the tail probability of W. That is, from Daniels (1987), we have

$$Pr(W \le w; \alpha) \simeq \Phi(z) + \phi(z)\{\frac{1}{z} - \frac{1}{\zeta}\}, \qquad (2.3)$$

where

$$z = \text{sgn}(\hat{\xi})\,[\,2\{\hat{\xi} w - K_W(\hat{\xi})\}\,]^{\frac{1}{2}},$$

$$\zeta = \hat{\xi}\{K''_W(\hat{\xi})\}^{\frac{1}{2}},$$

and w is the observed value from real data. Here, $\hat{\xi}$ is the solution of saddlepoint equation $K'_W(\xi) = w$ and $K'_W(\xi)$, $K''_W(\xi)$ are given by

$$K'_W(\xi) = \psi(n\alpha - \xi) - \psi(\alpha - \xi/n) - \log n,$$

$$K''_W(\xi) = \psi'(\alpha - \xi/n) - \psi'(n\alpha - \xi),$$

where $\psi(\cdot)$ and $\psi'(\cdot)$ are digamma and trigamma function respectively.

To construct the approximate confidence intervals, let $P_N(\alpha)$ be the right hand side of (2.3). From the approximation $P_N(\alpha)$, we can easily find the values α_L and α_U satisfying

$$P_N(\alpha_L) = \frac{\gamma}{2}, \qquad P_N(\alpha_U) = 1-\frac{\gamma}{2}, \tag{2.4}$$

respectively. Then (α_L, α_U) is the $100(1-\gamma)\%$ approximate confidence interval for the shape parameter of gamma distribution.

Example 1 in Section 3 shows the accuracy of this method. To obtain the exact value in Table 1, we need extensive tables of critical values. But, the suggested method is very accurate even for small sample size and easy to use for a given data set.

To compare this method with others, let us introduce Wong's (1992) which is known as the most accurate approximate method so far. He suggests a conditional method which also uses saddlepoint approximation. His method is as follows : Let $S_1 = \sum_{i=1}^n T_i$ and $S_2 = \sum_{i=1}^n \log T_i$. Then the joint density of (S_1, S_2) can be written as

$$f(s_1, s_2 ; \alpha, \beta) = \{\Gamma(\alpha)\beta^\alpha\}^{-n}\exp\{\alpha s_1 - s_2/\beta\}h_n(s_1, s_2), \tag{2.5}$$

where $h_n(s_1, s_2)$ is a marginal measure

$$\int\prod dt_i/t_i = \int h_n(s_1, s_2)\,ds_1\,ds_2.$$

Moreover, (2.5) can be factored into

$$f(s_1, s_2 ; \alpha, \beta) = f(s_1 ; \alpha, \beta) \times f(s_2 \mid s_1 ; \alpha),$$

where

$$f(s_1 ; \alpha, \beta) = \{\Gamma(n\alpha)\beta^{n\alpha}\}^{-1}\exp\{n\alpha\log s_1 - s_1/\beta\}/s_1$$

and

$$f(s_2 \mid s_1 ; \alpha) = \Gamma(n\alpha)\{\Gamma(\alpha)\}^{-n}\exp\{\alpha(s_2 - n\log s_1)\}h_n(s_1, s_2).$$

Note that the explicit form of $h_n(s_1, s_2)$ is hard to obtain in general.

Let us define the probability $P_W(\alpha)$ be the saddlepoint approximation which is given by Barndorff-Nielsen (1990) to the tail probability $Pr\{S_2 \le s_2 \mid S_1 = s_1; \alpha\}$. That is, $P_W(\alpha)$ is given by

$$\Phi(r) + \phi(r)\{\frac{1}{r} - \frac{1}{q}\}, \tag{2.6}$$

where

$$r = \mathrm{sgn}(\hat{\alpha} - \alpha)[2\{l_c(\hat{\alpha}) - l_c(\alpha)\}]^{\frac{1}{2}}, \tag{2.7}$$

$$q = (\hat{\alpha} - \alpha)\{j(\hat{\alpha})\}^{\frac{1}{2}}, \tag{2.8}$$

$$j(\hat{\alpha}) = j(\alpha)|_{\alpha=\hat{\alpha}} = -\partial^2 l_c(\alpha)/\partial\alpha^2|_{\alpha=\hat{\alpha}} \tag{2.9}$$

and $\hat{\alpha}$ satisfies $\partial l_c(\alpha)/\partial\alpha|_{\alpha=\hat{\alpha}} = 0$. Here, the conditional log-likelihood function $l_c(\alpha)$ is

$$\begin{aligned} l_c(\alpha) &= l_c(\alpha; s_2 \mid s_1) \\ &= \log\Gamma(n\alpha) - n\log\Gamma(\alpha) + \alpha(s_2 - n\log s_1), \end{aligned} \tag{2.10}$$

where (s_1, s_2) is available from the observed data. Analytically, $\hat{\alpha}$ and $j(\hat{\alpha})$ are difficult to obtain. A fine tabulation of (2.10) with step size δ can, however, be easily obtained and successive divided differences, $l_1(\alpha) = \{l_c(\alpha+\delta) - l_c(\alpha)\}/\delta$ and $l_2(\alpha) = \{l_1(\alpha) - l_1(\alpha-\delta)\}/\delta$, can be used to approximate $\partial_c l(\alpha)/\partial\alpha$ and $-j(\alpha)$. Thus $\hat{\alpha}$ and $j(\hat{\alpha})$ can be approximated numerically. Hence, by replacing (2.8) with

$$q \approx (\hat{\alpha} - \alpha)\{-l_2(\hat{\alpha})\}^{\frac{1}{2}}, \tag{2.11}$$

$P_W(\alpha)$ can be obtained by (2.6) with (2.7) and (2.11). To construct approximate confidence intervals, we can apply the same method as (2.4).

Finally, the differences between the proposed method and Wong's (1992) can be summarized as follows: Wong's (1992) method are based on conditional likelihood of S_2, given $S_1 = s_1$ and its derivatives up to order 2. But, the proposed method need only the well-known CGF of W and its derivatives. Also, the proposed method uses Schneider's (1978) algorithm to obtain the values of derivatives. This can avoids the difficulties of the choice of δ to obtain the derivatives of conditional likelihood in Wong's (1992).

3. Examples

Example 1. Consider an artificial data with $t_1=1$, $t_2=4$. Then the observed value of W is $\log(5/4)$ and the CGF of W is given by (2.2) with $n=2$. Table 1 records the exact $P_e(\alpha)$, $P_{BC}(\alpha)$ from the approximation by Barndorff-Nielsen and Cox (1979), $P_S(\alpha)$ from the approximation by Skovgaard (1987), $P_{JK}(\alpha)$ from the approximation by Jensen and Kristensen (1991), $P_W(\alpha)$ from the approximation by Wong (1992), and $P_N(\alpha)$ from the proposed method for $\alpha=.5(.5)8$. All the results except $P_N(\alpha)$ in Table 1 are cited from Wong (1992). We used Schneider's (1978) algorithm to obtain the value of trigamma function.

〈 Table 1 〉 Distribution Function for Example 1

α	$P_e(\alpha)$	$P_{BNC}(\alpha)$	$P_S(\alpha)$	$P_{JK}(\alpha)$	$P_W(\alpha)$	$P_N(\alpha)$
0.5	0.589	0.542	0.630	0.590	0.588	0.5877
1.0	0.400	0.389	0.432	0.399	0.399	0.3992
1.5	0.285	0.287	0.310	0.285	0.285	0.2852
2.0	0.208	0.219	0.228	0.209	0.209	0.2091
2.5	0.154	0.164	0.173	0.115	0.154	0.1557
3.0	0.116	0.126	0.129	0.117	0.116	0.1172
3.5	0.089	0.098	0.098	0.088	0.089	0.0889
4.0	0.067	0.076	0.075	0.067	0.067	0.0679
4.5	0.051	0.059	0.057	0.052	0.052	0.0520
5.0	0.039	0.046	0.044	0.040	0.040	0.0401
5.5	0.030	0.037	0.034	0.031	0.030	0.0309
6.0	0.023	0.029	0.026	0.024	0.024	0.0239
6.5	0.018	0.023	0.021	0.018	0.019	0.0186
7.0	0.014	0.018	0.016	0.014	0.014	0.0144
7.5	0.011	0.014	0.012	0.011	0.011	0.0113
8.0	0.009	0.011	0.010	0.009	0.009	0.0088

The proposed method outperformed both approximations by Barndorff-Nielsen and Cox (1979) and Skovgaard (1987), Wong (1992) and the proposed method are almost the same. Moreover, the simplicity of the proposed method is comparable to Wong (1992). A computer program, which is written in FORTRAN, has been developed to obtain the entries of $P_N(\alpha)$ in Table 1. A copy of this program is available from me on request.

Example 2. Pressure vessels constructed of fiber/epoxy composite materials wrapped around metal liners have lifetimes that may be modeled by a gamma distribution. The shape parameter depends on such factors as the applied pressure and the composite wall thickness. For suitably high pressures and/or thin wall, a DFR condition may occur. The following data which are cited from Keating et al.(1990) report the failure times (in hours) of 20 similarly constructed vessels subjected to a certain pressure :

274.000	28.500	1.700	20.800	871.000
363.000	1.311	1.661	236.000	828.000
458.000	290.000	54.900	175.000	1.787
970.000	0.750	1.278	776.000	126.000

Our attention is focussed on the following testing problem

$$H_0 : \alpha = 1 \qquad \text{versus} \qquad H_1 : \alpha < 1. \tag{3.1}$$

The observed value of W is given by $w = 1.0724$. Figure 1 plots $P_N(\alpha)$ obtaind by the proposed method ; any arbitrary significant level of α can be directly obtained from the plot. In particular, $P_N(1) = Pr\{W \le w ; \alpha = 1\} = 0.00716$, giving strong evidence against H_0. Moreover, the 90% upper confidence bound on α is 0.759. Keating et al.(1990) rejected H_0 at the 1% level of significance and reported the 90% upper confidence bound on α as 0.758 by Wilk interpolation and 0.760 by hybrid interpolation.

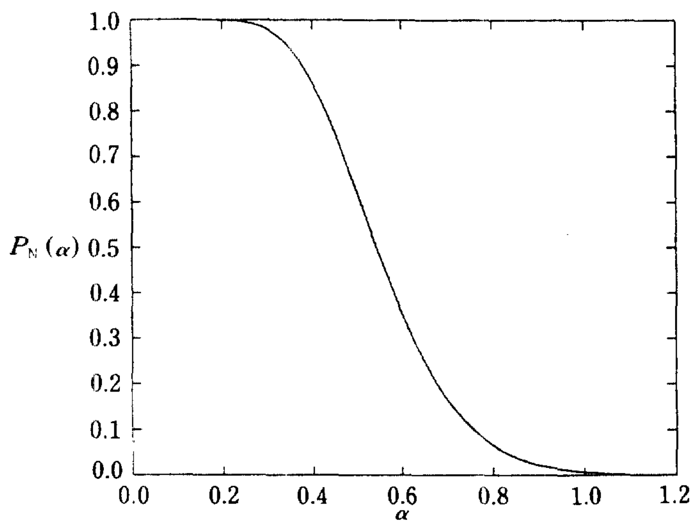

Figure 1. Confidence Distribution Function of α in Example 2

4. Conclusions

An unconditional approach for inferences on the shape parameter of a gamma distribution was provided in this paper. The proposed method require that the computer program be run for each data set. It is much simpler and easier to obtain approximations of any arbitrary significance level and confidence interval of α. The proposed numerical procedure are very accurate even for very small sample size.

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