■ 연구논문

Estimators of $Pr[X<Y]$ in Block and Basu's Bivariate Exponential Model*

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Abstract

The maximum likelihood estimator (M.L.E.) and the Bayes estimators of $Pr(X<Y)$ are derived when X and Y have a absolutely continuous bivariate exponential distribution in Block & Basu's model. The performances of M.L.E. are compared to those Bayes estimators for moderate sample size.

KEY WORDS: Block & Basu's absolutely continuous bivariate exponential distribution, Maximum Likelihood Estimator, Bayes Estimator

1. Introduction

According to the strength-stress model, a component fails when the stress exceeds its strength. The stress is a function of the environment to which the component is subjected and the strength of a component that is mass-produced depends on material properties, manufacturing procedures etc.

Let's consider the strength-stress model in the two component system. When we consider the dependence among the components that arise from common environmental shocks and stress, the assumption that the components of systems have underlying bivariate exponential distribution may be reasonable.

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Awad at al. (1981) derived the Maximum Likelihood Estimator (M.L.E.), moment type estimator and Mann-Whitney type estimator for $P=Pr[X<Y]$ in Marshall & Olkin's (1967) model. While Kim at al. (1989) derived the Bayes estimator for P in Marshell & Olkin's model with identical marginals and derived the M.L.E. and Bayes estimator for P in the Freund's (1961) model. Park (1990) derived the Bayes estimator for P in Marshall & Olkin's model with nonidentical marginals.

In this paper we derive the M.L.E. and Bayes Estimators of P when X and Y have a absolutely continuous bivariate exponential distribution in Block & Basu's (1974) model with identical marginals and nonidentical marginals and compare the performance of M.L.E. with Bayes estimators.

In Chapter 2, we study the M.L.E. for $Pr(X<Y)$ in the Block & Basu's bivariate exponential model. We study the Bayes estimators under the appropriate prior densities and quadratic loss function and weighted quadratic loss function for $Pr(X<Y)$ in the Block & Basu's bivariate exponential model in Chapter 3.

In Chatpter 4, bias and mean squared errors (M.S.E.) of each estimator are evaluated and compared and the conclusion is presented.

1.1 Notations

We introduce the following notations for convenience.

BVE = Marshall & Olkin's bivariate exponential distribution.

ACBVE = Block & Basu's absolutely continuous bivariate exponential distribution.

$\lambda \;= \lambda_0 + \lambda_1 + \lambda_2$.

$\underline{\lambda} \;= (\lambda_0, \lambda_1, \lambda_2)$.

$\underline{x} \;= (x_1, x_2, \cdots, x_n)$.

$\underline{y} \;= (y_1, y_2, \cdots, y_n)$.

t_1 = sum of first component failure times that failed first.

t_2 = sum of corresponding second component failure times.

t_1' = sum of those first component failure times that failed last.

t_2' = corresponding second component failure times.

$n_1 = \sum_{j=1}^{n} I(x_j<y_j)$.

$n_2 = \sum_{j=1}^{n} I(x_j>y_j)$.

$\hat{\lambda}_i^M$ = M.L.E. of λ_i.

$\hat{P}^M$ = M.L.E. of $Pr(X<Y)$.

$\hat{P}^B$ = Bayes Estimator of $Pr(X<Y)$.

$E_1(z) = \int_z^\infty \exp(-t)/t\,dt, \quad (|\arg z|<\pi)$.

1.2 Review of Bivariate Exponential Models

The random variables X and Y are said to follow the BVE if

$$\begin{aligned}\overline{F}(x, y) &= Pr(X > x, Y > y) \\ &= \exp\{-\lambda_1 x - \lambda_2 y - \lambda_0 \max(x, y)\}\end{aligned} \tag{1.1}$$

where $x > 0$, $y > 0$ and $\lambda_i \geq 0$ with at least one λ_i positive.

The BVE occupies an important place among bivariate life distributions in that it has the bivariate loss of memory property and its marginals have the loss of memory property. Since it is not absolutely continuous with respect to the usual Lébesque measure, we encounter the situations when it cannot be applied. Therefore an absolutely continuous distribution which is related to the BVE and has some of its properties is of interest. The ACBVE is a distribution with such characteristics.

We say that (X, Y) has the ACBVE if

$$\begin{aligned}\overline{F}_A(x, y) = &\frac{\lambda}{\lambda_1 + \lambda_2} \exp\{-\lambda_1 x - \lambda_2 y - \lambda_0 \max(x, y)\} - \\ &\frac{\lambda_0}{\lambda_1 + \lambda_2} \exp\{-\lambda \max(x, y)\},\end{aligned} \tag{1.2}$$

where $x > 0$, $y > 0$ and $\lambda_i > 0$, $i = 0, 1, 2$.

Equation (1.1) is decomposed into an ACBVE $\overline{F_A}(x, y)$ and a singular distribution

$$\overline{F}_S(x, y) = \exp\{-\lambda \max(x, y)\},$$

i.e.

$$\overline{F}(x, y) = \frac{\lambda_1 + \lambda_2}{\lambda} \overline{F}_A(x, y) + \frac{\lambda_0}{\lambda} \overline{F}_S(x, y). \tag{1.3}$$

Since the singular component of the BVE is exponential distribution with scale parameter $\lambda(e(\lambda))$, the marginals of the BVE are mixtures of the marginals of the ACBVE and $e(\lambda)$. So that the marginals of the ACBVE are weighted average of the marginals of the BVE and $e(\lambda)$. Therefore, the ACBVE does not have exponential marginals. But since $\overline{F}_A(x+t, y+t) = \overline{F}_A(x, y)\,\overline{F}_A(t, t)$ where x, y and $t > 0$, the ACBVE has the memoryless property.

The density of the ACBVE is as follows;

$$f_A(x, y) = \begin{cases} \dfrac{\lambda}{\lambda_1+\lambda_2}\lambda_1(\lambda_2+\lambda_0)\exp\{-\lambda_1 x-(\lambda_2+\lambda_0)y\}, & \text{if } x<y \\ \dfrac{\lambda}{\lambda_1+\lambda_2}\lambda_2(\lambda_1+\lambda_0)\exp\{-(\lambda_1+\lambda_0)x-\lambda_2 y\}, & \text{if } x>y, \end{cases} \tag{1.4}$$

where λ_0, λ_1 and $\lambda_2 > 0$. Therefore

$$Pr(X<Y) = \frac{\lambda}{\lambda_1+\lambda_2}[\exp\{-(\lambda_0+\lambda_2)t\}-\exp\{-\lambda t\}] \tag{1.5}$$

for any $t>0$.

As an another important bivariate exponential model, Freund (1961) introduced the distribution having the joint density

$$f_F(x, y) = \begin{cases} \alpha\beta'\exp\{-\beta' y-(\alpha+\beta-\beta')x\}, & \text{if } 0<x<y \\ \beta\alpha'\exp\{-\alpha' x-(\alpha+\beta-\alpha')y\}, & \text{if } 0<y<x \end{cases} \tag{1.6}$$

where α, β, α' and $\beta' > 0$.

The Freund's distribution does not have exponential marginals and does not allow the simultaneous failures. But this distribution has the loss of memory property. Freund showed that

$$\hat{P}^M = \frac{\hat{\alpha}^M}{\hat{\alpha}^M+\hat{\beta}^M-\hat{\beta}'^M}[\exp\{-\hat{\beta}'^M t\}-\exp\{-(\hat{\alpha}^M+\hat{\beta}^M)t\}] \tag{1.7}$$

maximizes

$$L(\alpha, \beta, \alpha', \beta' \mid \underline{x}, \underline{y}) = (\alpha\beta')^{n_1}(\beta\alpha')^{n_2} \cdot \exp\{-\beta' t_2-(\alpha+\beta-\beta')t_1-\alpha' t_1'-(\alpha+\beta-\alpha')t_2'\}, \tag{1.8}$$

where $\hat{\alpha}^M = \dfrac{n_1}{t_1+t_2'}$, $\hat{\beta}^M = \dfrac{n_2}{t_1+t_2'}$, $\hat{\alpha}'^M = \dfrac{n_2}{t_1'-t_2'}$ and $\hat{\beta}'^M = \dfrac{n_1}{t_2-t_1}$.

From Friday and Patil (1977), we know the relationship between parameters α, β, α', β' and λ_0, λ_1, λ_2. The relationship is as follows;

$$\alpha = \lambda_1(1+\lambda_0(\lambda_1+\lambda_2)^{-1}),$$

$$\beta = \lambda_2(1+\lambda_0(\lambda_1+\lambda_2)^{-1}),$$
$$\alpha' = \lambda_0+\lambda_1 \qquad (1.9)$$

and

$$\beta' = \lambda_0+\lambda_2.$$

If $\alpha' > \alpha$ and $\beta' > \beta$, then (1.6) can be expressed as (1.4). In the same manner, if $\alpha' > \alpha$ and $\beta' > \beta$, then

$$Pr(X<Y) = \int_t^\infty \int_0^t f_F(x, y)dxdy$$
$$= \int_t^\infty \int_0^t f_A(x, y)dxdy. \qquad (1.10)$$

2. M.L.E.s of $Pr(X<Y)$

2.1 In the case of identical marginals

Consider a two component parallel redundancy system with each component life times X and Y from ACBVE. In this case, we have

$$P = \frac{\lambda_0+2\lambda_1}{2\lambda_1}[\exp\{-(\lambda_0+\lambda_1)t\}-\exp\{-(\lambda_0+2\lambda_1)t\}] \qquad (2.1.1)$$

for any $t>0$.

The likelihood function is expressed as follows;

$$L(\lambda_0, \lambda_1 | \underline{x}, \underline{y}) = (\frac{1}{2}\lambda_0+\lambda_1)^n(\lambda_0+\lambda_1)^n \cdot$$
$$\exp\{-\lambda_1(t_1+t_1'+t_2+t_2')-\lambda_0(t_1'+t_2)\}. \qquad (2.1.2)$$

Differentiating the log of the likelihood function partially with respect to λ_0 and λ_1, and equating the derivatives to zeros, we obtain the following equations;

$$\frac{n}{\lambda_0+2\lambda_1} + \frac{n}{\lambda_0+\lambda_1} = t_1'+t_2$$

and

$$\frac{n}{\frac{1}{2}\lambda_0+\lambda_1} + \frac{n}{\lambda_0+\lambda_1} = t_1+t_1'+t_2+t_2'. \qquad (2.1.3)$$

Using the solution of (2.1.3), we obtain that M.L.E.s of λ_1 and λ_0 are given as follows;

$$\hat{\lambda}_1^M = \frac{n(t_1' + t_2 - 2t_1 - 2t_2')}{(t_1 + t_2')(t_1' + t_2 - t_1 - t_2')}$$

and

$$\hat{\lambda}_0^M = \frac{n(3t_1 + 3t_2' - t_1' - t_2)}{(t_1 + t_2')(t_1' + t_2 - t_1 - t_2')}. \tag{2.1.4}$$

By the invariance property of M.L.E., we obtain that M.L.E. of P is given as follows;

$$\hat{P}^M = \frac{\hat{\lambda}_0^M + 2\hat{\lambda}_1^M}{2\hat{\lambda}_1^M}\left[\exp\{-(\hat{\lambda}_0^M + \hat{\lambda}_1^M)t\} - \exp\{-(\hat{\lambda}_0^M + 2\hat{\lambda}_1^M)t\}\right] \tag{2.1.5}$$

for any $t > 0$.

2.2 In the case of nonidentical marginals

Consider a parallel system of two nonidentical components based on bivariate exponential distribution with nonidentical marginals in ACBVE. We have the likelihood function from Block & Basu. The likelihood function is as follows;

$$L(\underline{\lambda} \mid \underline{x}, \underline{y}) = \left(\frac{\lambda}{\lambda_1 + \lambda_2}\right)^n [\lambda_1(\lambda - \lambda_1)]^{n_1} [\lambda_2(\lambda - \lambda_2)]^{n_2} \cdot \exp\{-\lambda_1(t_1 - t_2) - \lambda_2(t_2' - t_1') - \lambda(t_2 + t_1')\}. \tag{2.2.1}$$

From (2.2.1), Block & Basu obtained the likelihood equations as follows;

$$\frac{n}{\lambda} + \frac{n_1}{\lambda - \lambda_1} + \frac{n_2}{\lambda - \lambda_2} = t_2 + t_1',$$

$$-\frac{n}{\lambda_1 + \lambda_2} + \frac{n_1}{\lambda_1} - \frac{n_1}{\lambda - \lambda_1} = t_1 - t_2$$

and

$$-\frac{n}{\lambda_1 + \lambda_2} + \frac{n_2}{\lambda_2} - \frac{n_2}{\lambda - \lambda_2} = t_2' - t_1'.$$

Since these equations are too complicate to solve, we obtain the $\hat{P}^M$ easily by using

(1.7) through (1.10) only when $\alpha > \alpha'$ and $\beta > \beta'$. M.L.E. of P is given as follows;

$$\hat{P}^{M} = \frac{a}{a+b-c}\left[\exp\{-ct\} - \exp\{-(a+b)t\}\right] \tag{2.2.2}$$

where $a = \dfrac{n_1}{t_2+t_2'}$, $b = \dfrac{n_2}{t_1+t_2'}$ and $c = \dfrac{n_1}{t_2-t_1}$.

3. Bayes Estimators of $Pr(X<Y)$

3.1 In the case of identical marginals

3.1.1 Uniform prior

We assume a quadratic loss function given by

$$\ell(P,\hat{P}) = (P-\hat{P})^2 \tag{3.1.1}$$

and the uniform prior distribution for (λ_0, λ_1) is as follows;

$$g(\lambda_0, \lambda_1) \propto g_0(\lambda_0) g_1(\lambda_1) \tag{3.1.2}$$

where

$$g_i(\lambda_i) = \frac{1}{c_i} I_{(0,c_i)}(\lambda_i), \quad \lambda_i > 0 \text{ and } c_i > 0, \ i = 0, 1.$$

From (2.1.2), (3.1.1) and (3.1.2), we derived the joint posterior distribution of (λ_0, λ_1) as follows;

$$\begin{aligned} \pi_1(\lambda_0, \lambda_1 \mid \underline{x}, \underline{y}) = K_1 \frac{1}{c_0 c_1} \sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{i}\binom{n}{j}\left(\frac{1}{2}\right)^{i} \lambda_0^{i+j} \lambda_1^{2n-i-j} \cdot \\ \exp\{-\lambda_1(t_1+t_1'+t_2+t_2') - \lambda_0(t_2+t_1')\} \cdot \\ I_{(0,c_0)}(\lambda_0) I_{(0,c_1)}(\lambda_1) \end{aligned} \tag{3.1.3}$$

where

$$\begin{aligned} K_1^{-1} = \frac{1}{c_0 c_1} \sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{i}\binom{n}{j}\left(\frac{1}{2}\right)^{i} \Big[\frac{\Gamma(i+j+1)}{(t_2+t_1')^{i+j+1}} \\ - \sum_{k=0}^{i+j} \frac{c_0^{\,i+j-k}}{(t_2+t_1')^{k+1}} \frac{\Gamma(i+j+1)}{\Gamma(i+j-k+1)} \exp\{-c_0(t_2+t_1')\} \Big] \cdot \end{aligned}$$

$$\left[\frac{\Gamma(2n-i-j+1)}{(t_1+t_1'+t_2+t_2')^{2n-i-j+1}}-\sum_{k=0}^{2n-i-j}\frac{c_1^{\,2n-i-j-k}}{(t_1+t_1'+t_2+t_2')^{k+1}}\cdot\frac{\Gamma(2n-i-j+1)}{\Gamma(2n-i-j-k+1)}\exp\{-c_1(t_1+t_1'+t_2+t_2')\}\right].$$

Then the Bayes estimator of P is as follows;

$$\hat{P}^B = K_1(P_1-P_2) \tag{3.1.4}$$

where

$$P_1=\frac{1}{c_0\,c_1}\sum_{i=0}^{n+1}\sum_{j=0}^{n}\binom{n+1}{i}\binom{n}{j}\left(\frac{1}{2}\right)^i\left[\frac{\Gamma(i+j+1)}{(t_1'+t_2+t)^{i+j+1}}-\sum_{k=0}^{i+j}\frac{c_0^{\,i+j-k}}{(t_1'+t_2+t)^{k+1}}\cdot\frac{\Gamma(i+j+1)}{\Gamma(i+j-k+1)}\exp\{-c_0(t_1'+t_2+t)\}\right]\left[\frac{\Gamma(2n-i-j+1)}{(t_1+t_1'+t_2+t_2'+t)^{2n-i-j+1}}-\sum_{k=0}^{2n-i-j}\frac{c_1^{\,2n-i-j-k}}{(t_1+t_1'+t_2+t_2'+t)^{k+1}}\frac{\Gamma(2n-i-j+1)}{\Gamma(2n-i-j-k+1)}\cdot\exp\{-c_1(t_1+t_1'+t_2+t_2'+t)\}\right]$$

and

$$P_2=\frac{1}{c_0\,c_1}\sum_{i=0}^{n-1}\sum_{j=0}^{n}\binom{n+1}{i}\binom{n}{j}\left(\frac{1}{2}\right)^i\left[\frac{\Gamma(i+j+1)}{(t_1'+t_2+t)^{i+j+1}}-\sum_{k=0}^{i+j}\frac{c_0^{\,i+j-k}}{(t_1'+t_2+t)^{k+1}}\cdot\frac{\Gamma(i+j+1)}{\Gamma(i+j-k+1)}\exp\{-c_0(t_1'+t_2+t)\}\right]\left[\frac{\Gamma(2n-i-j+1)}{(t_1+t_1'+t_2+t_2'+2t)^{2n-i-j+1}}-\sum_{k=0}^{2n-i-j}\frac{c_1^{\,2n-i-j-k}}{(t_1+t_1'+t_2+t_2'+2t)^{k+1}}\frac{\Gamma(2n-i-j+1)}{\Gamma(2n-i-j-k+1)}\cdot\exp\{-c_1(t_1+t_1'+t_2+t_2'+2t)\}\right].$$

3.1.2 Gamma prior

Now we consider a gamma prior distribution. The gamma prior distribution for (λ_0, λ_1) is as follows;

$$g(\lambda_0, \lambda_1) \propto g_0(\lambda_0)\,g_1(\lambda_1) \tag{3.1.5}$$

where

$$g_i(\lambda_i) = \lambda^{\alpha_i-1}\exp\{-\beta_i\,\lambda_i\},\ \ \lambda_i>0,\ \ \alpha_i>0\ \text{and}\ \beta_i>0,\ \ i=0,1.$$

From (2.1.2), (2.2.1.5) and (2.2.1.6), we derived the joint posterior distribution of (λ_0, λ_1) as follows;

$$\pi_2(\lambda_0, \lambda_1 \mid \underline{x}, \underline{y}) = K_2 \sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{i} \binom{n}{j} \left(\frac{1}{2}\right)^i \lambda_0^{i+j+\alpha_0-1} \lambda_1^{2n-i-j+\alpha_1-1} \cdot \exp\{-\lambda_1(t_1+t_1'+t_2+t_2'+\beta_1)-\lambda_0(t_1'+t_2+\beta_0)\} \quad (3.1.6)$$

where

$$K_2^{-1} = \sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{i} \binom{n}{j} \left(\frac{1}{2}\right)^i \frac{\Gamma(i+j+\alpha_0)}{(t_1'+t_2+\beta_0)^{i+j+\alpha_0}} \cdot \frac{\Gamma(2n-i-j+\alpha_1)}{(t_1+t_1'+t_2+t_2'+\beta_1)^{2n-i-j+\alpha_1}}.$$

Then the Bayes estimator of P is as follows;

$$\hat{P}^B = K_2 \sum_{i=0}^{n+1} \sum_{j=0}^{n} \binom{n+1}{i} \binom{n}{j} \left(\frac{1}{2}\right)^i \frac{\Gamma(i-j+\alpha_0)}{(t_1'+t_2+\beta_0+t)^{i-j+\alpha_0}} \cdot \frac{\Gamma(2n-i-j+\alpha_1)}{(t_1+t_1'+t_2+t_2'+\beta_1+t)^{2n-i-j+\alpha_1}} - K_2 \sum_{i=0}^{n+1} \sum_{j=0}^{n} \binom{n+1}{i} \binom{n}{j} \left(\frac{1}{2}\right)^i \cdot \frac{\Gamma(i-j+\alpha_0)}{(t_1'+t_2+\beta_0+t)^{i+j+\alpha_0}} \frac{\Gamma(2n-i-j+\alpha_1)}{(t_1+t_1'+t_2+t_2'+\beta_1+2t)^{2n-i-j+\alpha_1}}. \quad (3.1.7)$$

3.2 In the case of nonidentical marginals

3.2.1 Uniform prior

We assume a weighted quadratic loss function given by

$$\ell(P, \hat{P}) = w(\underline{\lambda})(P-\hat{P})^2 \quad (3.2.1)$$

where $w(\underline{\lambda})=(\lambda_1+\lambda_2)^{n+1}$ and the uniform prior distribution for $\underline{\lambda}$ is as follows;

$$g(\lambda_0, \lambda_1, \lambda_2) \propto g_i(\lambda_i) \quad (3.2.2)$$

where

$$g_i(\lambda_i) = \frac{1}{c_i} I_{(0,c_i)}(\lambda_i),\ \lambda_i>0 \text{ and } c_i>0,\ i=0, 1, 2.$$

From (2.2.1), (3.2.1) and (3.2.2), we derived the joint posterior distribution of $\underline{\lambda}$ as follows;

$$\pi_3(\underline{\lambda}|\underline{x},\underline{y})=K_3H_1\frac{1}{c_0c_1c_2}\left(\frac{\lambda}{\lambda_1+\lambda_2}\right)^n\lambda_1^{n_1}\lambda_2^{n_1}(\lambda_0+\lambda_2)^{n_1}(\lambda_0+\lambda_1)^{n_2}\cdot$$
$$\exp\{-\lambda_1(t_1+t_1')-\lambda_2(t_2+t_2')-\lambda_0(t_1'+t_2)\}\cdot \qquad (3.2.3)$$
$$I_{(0,c_0)}(\lambda_0)I_{(0,c_1)}(\lambda_1)I_{(0,c_2)}(\lambda_2)$$

where

$$K_3^{-1}=\frac{1}{c_0c_1c_2}\sum_{i=0}^{n}\sum_{j=0}^{n_1}\sum_{k=0}^{n_2}\sum_{l=0}^{n-j}\binom{n}{i}\binom{n_1}{j}\binom{n_2}{k}\binom{n-j}{l}\cdot$$
$$(-1)^{n-j-l}\left[\frac{\Gamma(i+j+k+1)}{(t_1'+t_2)^{i+j+k+1}}-\sum_{m=0}^{i+j+k}\frac{c_0^{\,i+j+k-m}}{(t_1'+t_2)^{m+1}}\cdot\right.$$
$$\left.\frac{\Gamma(i+j+k+1)}{\Gamma(i+j+k-m+1)}\exp\{-(t_1'+t_2)c_0\}\right]\left[\frac{\Gamma(2n-j-l-k+1)}{(t_1+t_1'-t_2-t_2')^{2n-j-l-k+1}}\right.$$
$$-\sum_{m=0}^{2n-j-l-k}\frac{\tau^{2n-j-l-k-m}}{(t_1+t_1'-t_2-t_2')^{m+1}}\frac{\Gamma(2n-j-l-k+1)}{\Gamma(2n-j-l-k-m+1)}\cdot$$
$$\left.\exp\{-(t_1+t_1'-t_2-t_2')\tau\}\right]$$

and

$$H_1^{-1}=\sum_{m=0}^{l-i}\frac{\tau^{l-i-m}}{(t_2+t_2')^{m+1}}\frac{\Gamma(l-i+1)}{\Gamma(l-i-m+1)}\cdot$$
$$\exp\{-(t_2+t_2')\tau\}-\sum_{m=0}^{l-i}\frac{(c_1+c_2)^{l-i-m}}{(t_2+t_2')^{m+1}}\cdot$$
$$\frac{\Gamma(l-i+1)}{\Gamma(l-i-m+1)}\exp\{-(c_1+c_2)(t_2+t_2')\},\quad \text{if } (l-i)\ge 0$$

$$H_1^{-1}=\sum_{m=1}^{l-i-1}\frac{(t_2+t_2')^{m-1}}{\tau^{i-l-m}}(-1)^m\frac{\Gamma(i-l-m)}{\Gamma(i-l)}\exp\{-(t_2+t_2')\tau\}$$
$$-\sum_{m=1}^{i-l-1}\frac{(t_2+t_2')^{m-1}}{(c_1+c_2)^{i-l-m}}(-1)^m\frac{\Gamma(i-l-m)}{\Gamma(i-l)}\exp\{-(c_1+c_2)(t_2+t_2')\}$$
$$+(-1)^{i-l-1}\frac{(t_2+t_2')^{i-l-1}}{\Gamma(i-l)}.$$

$$[E_1\{(t_2+t_2')\tau\}-E_1\{(c_1+c_2)(t_2+t_2')\}],\quad \text{if } (l-i)<0$$

for any $0<\tau<c_1+c_2$.

Then the Bayes estimator of P is as follows;

$$\hat{P}^B = Q(P_1 - P_2) \tag{3.2.4}$$

where

$$Q^{-1} = \sum_{i=0}^{n}\sum_{j=0}^{i+1}\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}\binom{n}{i}\binom{i+1}{j}\binom{n_1}{k}\binom{n_2}{l}\cdot$$

$$\Big[\frac{\Gamma(n-i+k+l+1)}{(t_1'+t_2)^{n-i+k+l+1}}-\sum_{m=0}^{n-i+k+l}\frac{c_0^{\,n-i+k+l-m}}{(t_1'+t_2)^{m+1}}\cdot$$

$$\frac{\Gamma(n-i+k+l+1)}{\Gamma(n-i+k+l-m+1)}\exp\{-c_0(t_1'+t_2)\}\Big]\Big[\frac{\Gamma(n+j-l+1)}{(t_1+t_1')^{n+j-l+1}}$$

$$-\sum_{m=0}^{n+j-l}\frac{c_1^{\,n+j-l-m}}{(t_1+t_1')^{m+1}}\frac{\Gamma(n+j-l+1)}{\Gamma(n+j-l-m+1)}\exp\{-c_1(t_1+t_1')\}\Big]\cdot$$

$$\Big[\frac{\Gamma(n+i-j-k+2)}{(t_2+t_2')^{n+i-j-k+2}}-\sum_{m=0}^{n+i-j-k+1}\frac{c_2^{\,n+i-j-k+1-m}}{(t_2+t_2')^{m+1}}\cdot$$

$$\frac{\Gamma(n+i-j-k+2)}{\Gamma(n+i-j-k-m+2)}\exp\{-c_2(t_2+t_2')\}\Big],$$

$$P_1 = \sum_{i=0}^{n+1}\sum_{j=0}^{i}\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}\binom{n+1}{i}\binom{i}{j}\binom{n_1}{k}\binom{n_2}{l}\cdot$$

$$\Big[\frac{\Gamma(n-i+k+l+2)}{(t_1'+t_2+t)^{n-i+k+l+2}}-\sum_{m=0}^{n-i+k+l+1}\frac{c_0^{\,n-i+k+l+1-m}}{(t_1'+t_2+t)^{m+1}}\cdot$$

$$\frac{\Gamma(n-i+k+l+2)}{\Gamma(n-i+k+l+2-m)}\exp\{-c_0(t_1'+t_2+t)\}\Big]\Big[\frac{\Gamma(n+j-l+1)}{(t_1+t_1')^{n+j-l+1}}$$

$$-\sum_{m=0}^{n+j-l}\frac{c_1^{\,n+j-l-m}}{(t_1+t_1')^{m+1}}\frac{\Gamma(n+j-l+1)}{\Gamma(n+j-l+1-m)}\exp\{-c_1(t_1+t_1')\}\Big]\cdot$$

$$\Big[\frac{\Gamma(n+i-j-k+1)}{(t_2+t_2'+t)^{n+i-j-k+1}}-\sum_{m=0}^{n+i-j-k}\frac{c_2^{\,n+i-j-k-m}}{(t_2+t_2'+t)^{m+1}}\cdot$$

$$\frac{\Gamma(n+i-j-k+1)}{\Gamma(n+i-j-k+1-m)}\exp\{-c_2(t_2+t_2'+t)\}\Big]$$

and

$$P_2 = \sum_{i=0}^{n+1}\sum_{j=0}^{i}\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}\binom{n+1}{i}\binom{i}{j}\binom{n_1}{k}\binom{n_2}{l}\cdot$$

$$\Big[\frac{\Gamma(n-i+k+l+2)}{(t_1'+t_2+t)^{n-i+k+l+2}} - \sum_{m=0}^{n-i+k+l+1}\frac{c_0^{\,n-i+k+l+1-m}}{(t_1'+t_2+t)^{m+1}}\cdot$$

$$\frac{\Gamma(n-i+k+l+2)}{\Gamma(n-i+k+l+2-m)}\exp\{-c_0(t_1'+t_2+t)\}\Big]\Big[\frac{\Gamma(n+j-l+1)}{(t_1+t_1'+t)^{n+j-l+1}}$$

$$-\sum_{m=0}^{n+j-l}\frac{c_1^{\,n+j-l-m}}{(t_1+t_1'+t)^{m+1}}\frac{\Gamma(n+j-l+1)}{\Gamma(n+j-l+1-m)}\cdot$$

$$\exp\{-c_1(t_1+t_1'+t)\}\Big]\Big[\frac{\Gamma(n+i-j-k+1)}{(t_2+t_2'+t)^{n+i-j-k+1}} - \sum_{m=0}^{n+i-j-k}\frac{c_2^{\,n+i-j-k-m}}{(t_2+t_2'+t)^{m+1}}\cdot$$

$$\frac{\Gamma(n+i-j-k+1)}{\Gamma(n+i-j-k+1-m)}\exp\{-c_2(t_2+t_2'+t)\}\Big].$$

3.2.2 Gamma prior

Now we consider a gamma prior distribution. The gamma prior distribution for $\underline{\lambda}$ is as follows;

$$g(\underline{\lambda}) \propto g_0(\lambda_0)g_1(\lambda_1)g_2(\lambda_2) \qquad (3.2.5)$$

where

$g_i(\lambda_i) = \lambda_i^{\alpha_i-1}\exp\{-\beta_i\lambda_i\}$, $\lambda_i>0$, $\alpha_i>0$ and $\beta_i>0$, $i=0, 1, 2$.

From (2.2.1), (3.2.1) and (3.2.5), we derived the joint posterior distribution of $\underline{\lambda}$ as follows;

$$\pi_4(\underline{\lambda}\mid\underline{x},\underline{y}) = K_4 H_2\Big(\frac{\lambda}{\lambda_1+\lambda_2}\Big)^n \lambda_0^{\alpha_0-1}\lambda_1^{n_1+\alpha_1-1}\lambda_2^{n_2+\alpha_2-1}\cdot$$

$$(\lambda_0+\lambda_2)^{n_1}(\lambda_0+\lambda_1)^{n_2}\cdot \qquad (3.2.6)$$

$$\exp\{-\lambda_1(t_1+t_1'+\beta_1)-\lambda_2(t_2+t_2'+\beta_2)-\lambda_0(t_1'+t_2+\beta_0)\}$$

where

$$K_4^{-1} = \sum_{i=0}^{n}\sum_{j=0}^{n_1}\sum_{k=0}^{n_2}\sum_{l=0}^{n-j+\alpha_2-1}\binom{n}{i}\binom{n_1}{j}\binom{n_2}{k}\binom{n-j+\alpha_2-1}{l}(-1)^{n-j-l+\alpha_2-1}\cdot$$

$$\frac{\Gamma(i+j+k+\alpha_0)}{t_0^{\,i+j+k+\alpha_0}}\Bigg[\frac{\Gamma(2n-j-l-k+\alpha_1+\alpha_2-1)}{(t_1+t_1'-t_2-t_2')^{2n-j-l-k+\alpha_1+\alpha_2-1}}$$

$$-\sum_{m=0}^{2n-j-l-k+\alpha_1+\alpha_2-2}\frac{\tau^{2n-j-l-k-m+\alpha_1+\alpha_2-2}}{(t_1+t_1'-t_2-t_2')^{m+1}}\cdot$$

$$\frac{\Gamma(2n-j-l-k+\alpha_1+\alpha_2-1)}{\Gamma(2n-j-l-k-m+\alpha_1+\alpha_2-1)}\exp\{-(t_1+t_1'-t_2-t_2')\tau\}\Bigg]$$

and

$$H_2^{-1}=\sum_{m=0}^{l-i}\frac{\tau^{l-i-m}}{(t_2+t_2')^{m+1}}\frac{\Gamma(l-i+1)}{\Gamma(l-i-m+1)}\exp\{-(t_2+t_2')\tau\},\quad \text{if } (l-i)\geq 0$$

$$H_2^{-1}=\sum_{m=1}^{i-l-1}\frac{(-1)^{m-1}}{\tau^{i-l-m}}(t_2+t_2')^{m-1}\frac{\Gamma(i-l-m)}{\Gamma(i-l)}\exp\{-(t_2+t_2')\tau\}$$

$$+(-1)^{i-l-1}\frac{(t_2+t_2')^{i-l-1}}{\Gamma(i-l)}E_1\{(t_2+t_2')\tau\},\quad \text{if } (l-i)<0$$

for any $\tau>0$.
Then the Bayes estimator of P is as follows;

$$\hat{P}^{B}=Q(P_1-P_2) \tag{3.2.7}$$

where

$$Q^{-1}=\sum_{i=0}^{n}\sum_{j=0}^{i-1}\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}\binom{n}{i}\binom{i+1}{j}\binom{n_1}{k}\binom{n_2}{l}\cdot$$

$$\frac{\Gamma(n-i+k+l+\alpha_0)}{(t_1'+t_2+\beta_0)^{n-i+k+l+\alpha_0}}\;\frac{\Gamma(n+j-l+\alpha_1)}{(t_1+t_1'+\beta_1)^{n+j-l+\alpha_1}}\cdot$$

$$\frac{\Gamma(n+i-j-k+\alpha_2+1)}{(t_2+t_2'+\beta_2)^{n+i-j-k+\alpha_2+1}},$$

$$P_1=\sum_{i=0}^{n+1}\sum_{j=0}^{i}\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}\binom{n+1}{i}\binom{i}{j}\binom{n_1}{k}\binom{n_2}{l}\cdot$$

$$\frac{\Gamma(n-i+k+l+\alpha_0+1)}{(t_1'+t_2+\beta_0+t)^{n-i+k+l+\alpha_0+1}}\;\frac{\Gamma(n+j-l+\alpha_1)}{(t_1+t_1'+\beta_1)^{n+j-l+\alpha_1}}\cdot$$

$$\frac{\Gamma(n+i-j-k+\alpha_2)}{(t_2+t_2'+\beta_2+t)^{n+i-j-k+\alpha_2}}$$

and

$$P_2 = \sum_{i=0}^{n+1}\sum_{j=0}^{i}\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}\binom{n+1}{i}\binom{i}{j}\binom{n_1}{k}\binom{n_2}{l}\cdot$$

$$\frac{\Gamma(n-i+k+l+\alpha_0+1)}{(t_1'+t_2+\beta_0+t)^{n-i+k+l+\alpha_0+1}}\;\frac{\Gamma(n+j-l+\alpha_1)}{(t_1+t_1'+\beta_1+t)^{n+j-l+\alpha_1}}\cdot$$

$$\frac{\Gamma(n+i-j-k+\alpha_2)}{(t_2+t_2'+\beta_2+t)^{n+i-j-k+\alpha_2}}.$$

4. Empirical Comparison and Conclusion

Though we obtain the M.L.E. and Bayes estimators of $P=Pr(X<Y)$, the exact distributions of such estimators are very difficult to derive analytically. Thus, through the Monte Carlo simulation, we compared the relative performances of the M.L.E. and Bayes estimators of P. For the generation of two dependent exponential random variables of Block & Basu's model, we use the method proposed by Friday and Patil (1977).

4.1 In the case with identical marginals

We consider the bivariate exponential distribution with identical marginals, $\lambda_1=\lambda_2$, in Block & Basu's model. The estimates of the bias and M.S.E. were obtained from 1,500 trials and $n=5$ under the uniform prior distribution with $c_0=c_1=1$ and under the gamma prior distribution with $\alpha_0=\alpha_1=\beta_0=\beta_1=1$. From tables 1 and 2, we observe the following facts for the ACBVE with identical marginals ;

(1) The M.L.E. of $Pr(X<Y)$ performs better than the Bayes estimators with respect to bias when the true values of P is not too small.

(2) The bias of Bayes estimator with uniform prior distribution is less than the bias of Bayes estimator with gamma prior distribution when the M.L.E. of P don't perform better than the Bayes estimators with respect to bias.

(3) The M.S.E. of Bayes estimators for P is less than that of M.L.E.

(4) The M.S.E. of Bayes estimator with gamma prior distribution is less than Bayes estimator with uniform prior distribution in moderate value of P.

(5) As $Pr(X<Y)$ increases, the M.S.E. of Bayes estimator with uniform prior distribution increases.

(6) For fixed λ_0, as λ_1 increases, the bias of Bayes estimators decreases initially, and then increases.

(7) For fixed λ_1, as λ_0 increases, the Bias of Bayes estimators decreases initially, and then increases.

4.2 In the case with nonidentical marginals

We also consider the ACBVE with nonidentical marginals. The estimates of the bias and M.S.E. were obtained from 1,000 trials and $n=5$ under the uniform prior distribution with $c_0=c_1=c_2=1$ and under the gamma prior distribution with $\alpha_0=\alpha_1=\alpha_2=\beta_0=\beta_1=\beta_2=1$. From tables 3 and 4, we observe the following facts for the ACBVE with nonidentical marginals;

(1) The M.L.E. of $Pr(X<Y)$ performs better than the Bayes estimators of P with respect to bias in most cases.

(2) The bias of Bayes estimator of P with gamma prior decreases as λ_{i_1} increases for fixed λ_{i_2}, λ_{i_3} where $\{i_1, i_2, i_3\}$ is a permutation of $\{0, 1, 2\}$.

(3) The Bayes estimators of P performs better than the M.L.E. of P with respect to M.S.E. in most cases.

(4) The M.S.E. of Bayes estimator of P with uniform prior distribution is less than the M.S.E. of Bayes estimator of P with gamma prior distribution.

(5) The M.S.E. of Bayes estimator of P with gamma prior decreases as λ_2 increases for fixed $\lambda_0=1$, $0<\lambda_1\leq 1$ and $\lambda_1=1$, $0<\lambda_0\leq 1$.

(6) The M.S.E. of Bayes estimator of P with gamma prior increases as λ_0 increases for fixed $\lambda_2=1$, $0<\lambda_1\leq 1$.

〈 Table 1 〉 Estimates of $Pr(X<Y)$ in Block & Basu's Bivariate Exponential identical case ($t=0.22$), $n=5$

$(\lambda_0$	$\lambda_1)$	$Pr(X<Y)$	MSE			BIAS		
			MLE	BE(unif)	BE(gamma)	MLE	BE(unif)	BE(gamma)
0.25	0.25	0.0719	0.0017	0.0004	0.0005	0.0130	0.0066	0.0092
0.25	0.50	0.1104	0.0025	0.0005	0.0004	0.0149	−0.0004	0.0020
0.25	0.75	0.1424	0.0035	0.0007	0.0004	0.0133	−0.0088	−0.0105
0.25	1.00	0.1688	0.0038	0.0008	0.0008	0.0139	−0.0168	−0.0246
0.50	0.25	0.0907	0.0022	0.0006	0.0007	0.0128	0.0092	0.0124
0.50	0.50	0.1254	0.0032	0.0006	0.0004	0.0157	0.0011	0.0015
0.50	0.75	0.1540	0.0034	0.0006	0.0004	0.0122	−0.0077	−0.0128
0.50	1.00	0.1775	0.0038	0.0008	0.0009	0.0116	−0.0157	−0.0275
0.75	0.25	0.1074	0.0024	0.0007	0.0006	0.0124	0.0098	0.0123
0.75	0.50	0.1385	0.0035	0.0006	0.0003	0.0135	0.0014	−0.0014
0.75	0.75	0.1640	0.0037	0.0006	0.0004	0.0125	−0.0072	−0.0164
0.75	1.00	0.1848	0.0037	0.0007	0.0010	0.0179	−0.0153	−0.0307
1.00	0.25	0.1219	0.0030	0.0007	0.0004	0.0138	0.0103	0.0101
1.00	0.50	0.1498	0.0037	0.0006	0.0002	0.0116	0.0010	−0.0053
1.00	0.75	0.1725	0.0036	0.0006	0.0005	0.0116	−0.0069	−0.0201
1.00	1.00	0.1908	0.0040	0.0006	0.0012	0.0051	−0.0150	−0.0334

〈 Table 2 〉 Estimates of $Pr(X<Y)$ in Block & Basu's Bivariate Exponential identical case ($t=0.44$), $n=5$

$(\lambda_0$	$\lambda_1)$	$Pr(X<Y)$	MSE			BIAS		
			MLE	BE(unif)	BE(gamma)	MLE	BE(unif)	BE(gamma)
0.25	0.25	0.1254	0.0033	0.0007	0.0008	0.0157	0.0039	0.0078
0.25	0.50	0.1775	0.0038	0.0007	0.0006	0.0099	−0.0128	−0.0065
0.25	0.75	0.2112	0.0040	0.0011	0.0007	0.0052	−0.0268	−0.0199
0.25	1.00	0.2310	0.0039	0.0016	0.0010	−0.0056	−0.0374	−0.0290
0.50	0.25	0.1498	0.0036	0.0006	0.0008	0.0115	0.0038	0.0101
0.50	0.50	0.1908	0.0038	0.0006	0.0004	0.0043	−0.0131	−0.0051
0.50	0.75	0.2162	0.0036	0.0009	0.0005	−0.0013	−0.0257	−0.0169
0.50	1.00	0.2300	0.0037	0.0013	0.0006	−0.0090	−0.0342	−0.0233
0.75	0.25	0.1677	0.0032	0.0005	0.0006	0.0056	0.0017	0.0103
0.75	0.50	0.1994	0.0037	0.0005	0.0003	−0.0008	−0.0137	−0.0039
0.75	0.75	0.2179	0.0036	0.0008	0.0003	−0.0058	−0.0243	−0.0132
0.75	1.00	0.2266	0.0036	0.0011	0.0003	−0.0142	−0.0311	−0.0170
1.00	0.25	0.1803	0.0036	0.0004	0.0005	0.0045	−0.0004	0.0097
1.00	0.50	0.2041	0.0037	0.0004	0.0002	−0.0032	−0.0137	−0.0021
1.00	0.75	0.2169	0.0037	0.0007	0.0001	−0.0090	−0.0224	−0.0086
1.00	1.00	0.2215	0.0039	0.0010	0.0001	−0.0160	−0.0274	−0.0097

〈 Table 3 〉 Estimates of $Pr(X<Y)$ in Block & Basu's Bivariate Exponential nonidentical case ($t=0.44$), $n=5$

(λ_0	λ_1	λ_2)	$Pr(X<Y)$	MSE			BIAS		
				MLE	BE(unif)	BE(gamma)	MLE	BE(unif)	BE(gamma)
1.00	0.25	0.25	0.1803	0.0120	0.0018	0.0157	−0.0042	0.0360	0.0682
1.00	0.25	0.50	0.1256	0.0091	0.0062	0.0104	−0.0118	0.0757	0.0453
1.00	0.25	0.75	0.0965	0.0069	0.0104	0.0077	−0.0124	0.0995	0.0415
1.00	0.25	1.00	0.0778	0.0059	0.0135	0.0059	−0.0123	0.1145	0.0351
1.00	0.50	0.25	0.2659	0.0170	0.0014	0.0183	0.0053	−0.0320	0.0680
1.00	0.50	0.50	0.2041	0.0134	0.0005	0.0120	−0.0209	0.0134	0.0310
1.00	0.50	0.75	0.1646	0.0120	0.0025	0.0106	−0.0156	0.0466	0.0284
1.00	0.50	1.00	0.1365	0.0098	0.0052	0.0080	−0.0179	0.0703	0.0217
1.00	0.75	0.25	0.3243	0.0166	0.0075	0.0148	−0.0052	−0.0845	0.0468
1.00	0.75	0.50	0.2615	0.0173	0.0014	0.0151	−0.0079	−0.0341	0.0340
1.00	0.75	0.75	0.2169	0.0144	0.0002	0.0118	−0.0164	0.0030	0.0210
1.00	0.75	1.00	0.1832	0.0137	0.0012	0.0103	−0.0152	0.0322	0.0135
0.25	1.00	0.25	0.3428	0.0195	0.0070	0.0309	0.0175	−0.0781	0.1201
0.25	1.00	0.50	0.2986	0.0184	0.0042	0.0234	0.0073	−0.0592	0.0735
0.25	1.00	0.75	0.2620	0.0169	0.0018	0.0178	−0.0064	−0.0365	0.0350
0.25	1.00	1.00	0.2310	0.0160	0.0006	0.0145	−0.0164	−0.0127	0.0167
0.50	1.00	0.25	0.3583	0.0173	0.0117	0.0199	−0.0016	−0.1053	0.0739

〈 Table 4 〉 Estimates of $Pr(X<Y)$ in Block & Basu's Bivariate Exponential nonidentical case ($t=0.44$), $n=5$

(λ_0	λ_1	λ_2)	$Pr(X<Y)$	MSE			BIAS		
				MLE	BE(unif)	BE(gamma)	MLE	BE(unif)	BE(gamma)
0.50	1.00	0.50	0.3057	0.0177	0.0054	0.0176	−0.0078	−0.0708	0.0440
0.50	1.00	0.75	0.2641	0.0181	0.0019	0.0161	−0.0203	−0.0393	0.0177
0.50	1.00	1.00	0.2300	0.0158	0.0004	0.0128	−0.0252	−0.0117	0.0044
0.75	1.00	0.25	0.3668	0.0185	0.0144	0.0170	0.0005	−0.1184	0.0576
0.75	1.00	0.50	0.3081	0.0210	0.0058	0.0175	−0.0051	−0.0742	0.0341
0.75	1.00	0.75	0.2628	0.0177	0.0017	0.0136	−0.0212	−0.0380	0.0129
0.75	1.00	1.00	0.2266	0.0172	0.0003	0.0124	−0.0268	−0.0074	0.0004
0.25	0.25	1.00	0.0721	0.0052	0.0074	0.0037	−0.0081	0.0756	0.0140
0.25	0.50	1.00	0.1329	0.0105	0.0042	0.0092	−0.0095	0.0564	0.0192
0.25	0.75	1.00	0.1853	0.0138	0.0011	0.0133	−0.0118	0.0234	0.0192
0.50	0.25	1.00	0.0754	0.0052	0.0105	0.0054	−0.0081	0.0973	0.0290
0.50	0.50	1.00	0.1361	0.0107	0.0046	0.0099	−0.0085	0.0630	0.0254
0.50	0.75	1.00	0.1868	0.0146	0.0011	0.0125	−0.0064	0.0259	0.0231
0.75	0.25	1.00	0.0772	0.0058	0.0122	0.0059	−0.0100	0.1074	0.0330
0.75	0.50	1.00	0.1372	0.0112	0.0050	0.0095	−0.0102	0.0676	0.0258
0.75	0.75	1.00	0.1859	0.0145	0.0011	0.0116	−0.0113	0.0286	0.0201
1.00	1.00	1.00	0.2215	0.0159	0.0002	0.0117	−0.0220	−0.0005	0.0089

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