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ON 3-DIMENSIONAL ALMOST CONTACT METRIC MANIFOLDS*

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The various structures on almost contact metric manifolds have been studied by many authors([2],[5],[6],[9],[11]). The purpose of the present paper is to study the structures on 3-dimensional almost contact metric manifolds.

1. Preliminaries

Let M be a $(2n+1)$-dimensional differentiable manifold of cass C^∞ covered by a system of coordinate neighborhoods $\{U; x^h\}$ in which there are given a tensor field ${\phi_i}^h$ of type (1,1), a vector field ξ^h and a 1-form η_i satisfying

$$\phi_j^i \phi_i^h = -\delta_j^h + \eta_j \xi^h, \phi_i^h \xi^i = 0, \eta_i \phi_j^i = 0, \eta_i \xi^i = 1, \tag{1.1}$$

where the indices $h, i, j \dots$ run over the range $\{1, 2, \dots, 2n+1\}$. Such a set of a tensor field ϕ of type (1,1), a vector ξ and a 1-form η is called an *almost contact structure* and a manifold with an almost contact structure an *almost contact manifold.* If, in an almost contact manifold, there is given a Riemannian metric g_{ji} such that

$$g_{ts} \phi_j^t \phi_i^s = g_{ji} - \eta_j \eta i, \eta_i = g_{ih} \xi^h, \tag{1.2}$$

then the almost contact structure is said to be *metric* and the manifold is called an *almost contact metric manifold*([1]).

An almost contact structure (ϕ, ξ, η) on M is said to be *normal* if

$$N_{ji}^h + (\partial_j \eta_i - \partial_i \eta_j)\xi^h = 0 \text{ or } [N(x,y) = [\phi,\phi](x,y) + (d\eta)(x,y) = 0],$$

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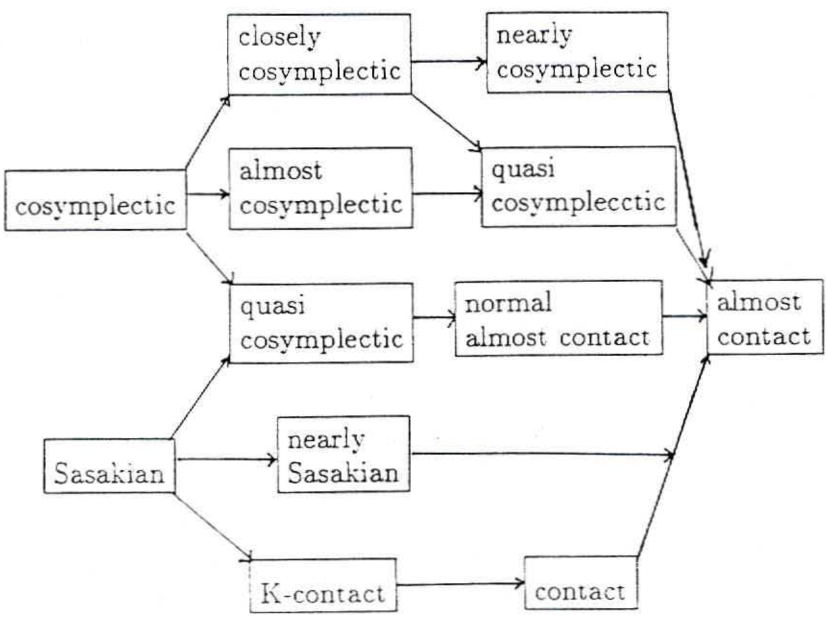
where

$$N_{ji}^h = \phi_j^t \partial_t \phi_i^h - \phi_i^t \partial_t \phi_j^h - (\partial_j \phi_i^t - \partial_i \phi_j^t)\phi_t^h$$

is the Nijenhuis tensor formed with ϕ_i^h and $\partial_j = \partial/\partial x^j$. We denote ∇ the covariant differentiation with respect to the Riemannian connection of g and denote $\phi_{ji} = \phi_j^h g_{hi}$. An almost contact metric structure (ϕ, ξ, η, g) on M is said to be

(a) contact ([1]), if $\phi_{ji} = \frac{1}{2}(\partial_j \eta_i - \partial_i \eta_j)$,
(b) K-contact ([1]), if $\nabla_i \eta_j = \phi_{ij}$,
(c) nearly Sasakian ([5]), if $\nabla_k \phi_j^h + \nabla_j \phi_k^h = -2g_{kj}\xi^h + \delta_k^h \eta_j + \delta_j^h \eta_k$,
(d) quasi Sasakian ([2]), if ϕ_{ji} is closed and (ϕ, ξ, η) is normal,
(e) Sasakian ([1]), if $\phi_{ji} = \frac{1}{2}(\partial_j \eta_i - \partial_i \eta_j)$ and (ϕ, ξ, η) is normal,
(f) nearly cosymplectic ([1],[3]), if ϕ_j^h is Killing,
(g) quasi cosymplectic ([7]), if $\nabla_k \phi_{ji} + \phi_k^t \phi_j^s \nabla_t \phi_{si} - \eta_j \phi_k^t \nabla_t \eta_i = 0$,
(h) closely cosymplectic ([4]), if ϕ_j^h is Killing and η_i is closed,
(i) almost cosymplectic ([9]), if ϕ_{ji} and η_j are closed,
(j) cosymplectic ([1]), if ϕ_{ji} and η_j are closed and (ϕ, ξ, η) is normal.

We note the following schematic array of structures ([8]).



If we put

$$E_{kjih} = \phi_{kj}\phi_{ih} - (g_{ki} - \eta_k \eta_i)(g_{jh} - \eta_j \eta_h) + (g_{ji} - \eta_j \eta_i)(g_{kh} - \eta_k \eta_h),$$

then we have $E_{kjih}E^{kjih} = 12n(n-1)$.

In a 3-dimensional almost contact metric manifold, E_{kjih} is a zero tensor.

Thus we have

Lemma 1. *In a 3-dimensional almost contact metric manifold, E_{kjih} vanishes identically, that is,*

$$(1.3)\quad \begin{aligned} \phi_{kj}\phi_{ih} &= (g_{ki}-\eta_k\eta_i)(g_{jh}-\eta_j\eta_h)-(g_{ji}-\eta_j\eta_i)(g_{kh}-\eta_k\eta_h). \\ &= \Upsilon_{ki}\Upsilon_{jh}-\Upsilon_{ji}\Upsilon_{kh}, \end{aligned}$$

where $\Upsilon_{ji} = g_{ji} - \eta_j\eta_i$.

On the other hand, it is well known that the conformal curvature tensor of Weyl vanishes identically in a 3-dimensional Riemannian manifold. Therefore the curvature tensor $K_{kji}{}^h$ of a 3-dimensional almost contact metric manifold M is given by

$$(1.4)\quad K_{kji}{}^h = -K_{ki}\delta_j^h + K_{ji}\delta_k^h - g_{ki}K_j^h + g_{ji}K_k^h + \frac{K}{2}(g_{ki}\delta_j^h - g_{ji}\delta_k^h),$$

where K_{ji} and K are the Ricci tensor and the scalar curvature of the manifold respectively.

Differentiating (1.3) covariantly, we obtain, in a 3-dimensional almost contact metric manifold,

$$\begin{aligned} \phi_{ih}\nabla_e\phi_{kj} + \phi_{kj}\nabla_e\phi_{ih} &= -(\eta_i\nabla_e\eta_k + \eta_k\nabla_e\eta_i)\Upsilon_{jh} - \Upsilon_{ki}(\eta_h\nabla_e\eta_j + \eta_j\nabla_e\eta_h) \\ &\quad +(\eta_i\nabla_e\eta_j + \eta_j\nabla_e\eta_i)\Upsilon_{kh} + \Upsilon_{ji}(\eta_h\nabla_e\eta_k + \eta_k\nabla_e\eta_h) \end{aligned}$$

Transvecting this equation with ϕ^{kj} and using $\phi^{kj}\phi_{kj} = 2$, we have

$$(1.5)\quad \nabla_e\phi_{ih} = (\nabla_e\eta_t)\phi_h^t\eta_i - (\nabla_e\eta_t)\phi_i^t\eta_h.$$

2. 3-dimensional K-contact manifolds

Let M be a 3-dimensional K-contact manifold. Then we have

$$(2.1)\quad \nabla_i\xi^h = \phi_i^h.$$

The equation (2.1) shows that ξ^h is a Killing vector field. Hence we have

$$(2.2)\quad \nabla_j\phi_i^h + K_{tji}{}^h\xi^t = 0.$$

It is well known ([1]) that on a 3-dimensional K-contact manifold the Ricci tensor satisfies

$$K_{jt}\xi^t = 2\eta_j. \tag{2.3}$$

Differentiating $\phi_i^h\eta_h = 0$ covariantly and taking account of (2.1) and (2.2), we obtain

$$K_{tji}{}^h\xi^t\eta_h = g_{ji} - \eta_j\eta_i. \tag{2.4}$$

Transvecting (1.4) with $\xi^k\eta_h$ and taking account of (2.3) and (2.4), we have

$$K_{ji} = (\frac{K}{2} - 1)g_{ji} + (3 - \frac{K}{2})\eta_j\eta_i. \tag{2.5}$$

Substituting (2.5) into (1.4), we obtain

$$\begin{aligned} K_{kji}{}^h &= (2 - \frac{K}{2})(g_{ki}\delta_j^h - g_{ji}\delta_k^h) \\ &\quad +(\frac{K}{2} - 3)[(\eta_k\delta_j^h - \eta_j\delta_k^h)\eta_i + (g_{ki}\eta_j - g_{ji}\eta_k)\xi^h]. \end{aligned} \tag{2.6}$$

Transvecting (2.6) with ξ^k and taking account of (2.2), we obtain

$$\nabla_j\phi_i^h = \delta_j^h\eta_i - g_{ji}\xi^h, \tag{2.7}$$

which shows that M is a Sasakian manifold.

Thus we have the following

Theorem 2. *A 3-dimensional K-contact manifold is a Sasakian manifold.*

Remark. S.Tanno also showed the same result in [12,13].

3. 3-dimensional nearly Sasakian manifolds

Let M be a 3-dimensional nearly Sasakian manifold. Then we have

$$\nabla_k\phi_{ji} + \nabla_j\phi_{ki} = -2g_{kj}\eta_i + g_{ki}\eta_j + g_{ji}\eta_k. \tag{3.1}$$

For a nearly Sasakian manifold the vector field ξ^h is Killing([5]), that is,

$$\nabla_j\eta_i + \nabla_i\eta_j = 0. \tag{3.2}$$

We define the tensor field H_{ji} by putting

$$\nabla_j\eta_i = \phi_{ji} + H_{ji}. \tag{3.3}$$

From the skew-symmetry of ϕ_{ji} and (3.2), it follows that H_{ji} is skew-symmetric. Set $H_j^i = H_{ja}g^{ai}$. Then we have the following equations [10].

$$H_{jt}\phi_i^t + H_{it}\phi_j^t = 0, \tag{3.4}$$

$$H_{jt}\xi^t = 0, \tag{3.5}$$

$$K_{tjih}\xi^t = -\nabla_j\phi_{ih} - \nabla_j H_{ih} = (g_{ji} + H_{jt}H_i^t)\eta^h - (g_{jh} + H_{jt}H_h^t)\eta_i. \tag{3.6}$$

On the other hand, transvecting (1.3) with ϕ_m^k, we obtain

$$(g_{ji} - \eta_j\eta_i)\phi_{mh} - (g_{jh} - \eta_j\eta_h)\phi_{mi} = (g_{mj} - \eta_m\eta_j)\phi_{ih}. \tag{3.7}$$

Transvecting (3.7) with H_l^h and taking account of (3.5), we have

$$(g_{ji} - \eta_j\eta_i)H_{lt}\phi_m^t - H_{lj}\phi_{mi} = (g_{mj} - \eta_m\eta_j)H_{lt}\phi_i^t. \tag{3.8}$$

Taking the symmetric part of (3.8) with respect to l and m and using (3.4), we can find

$$-H_{lj}\phi_{mi} - H_{mj}\phi_{li} = (g_{mj} - \eta_m\eta_j)H_{lt}\phi_i^t + (g_{lj} - \eta_l\eta_j)H_{mt}\phi_i^t. \tag{3.9}$$

Transvecting (3.9) with ϕ_p^i and taking account of (3.5), we obtain

$$\begin{aligned} & H_{lj}(g_{pm} - \eta_p\eta_m) + H_{mj}(g_{pl} - \eta_p\eta_l) \\ = \; & H_{lp}(g_{jm} - \eta_j\eta_m) + H_{mp}(g_{lj} - \eta_l\eta_j). \end{aligned} \tag{3.10}$$

Transvecting (3.10) with g^{mj} and taking account of (3.5) and $H_t^t = 0$, we have $H_{lp} = 0$. Hence (3.3) and (3.6) show that

$$\nabla_j\eta_i = \phi_{ji}, \nabla_k\phi_{ji} = -g_{kj}\eta_i + g_{ki}\eta_j.$$

Therefore M is a Sasakian manifold. Thus we have

Theorem 3. *A 3-dimensional nearly Sasakian manifold is a Sasakian manifold.*

4. 3-dimensional nearly cosymplectic manifolds

Suppose that M is a 3-dimensional nearly cosymplectic manifold. Then ϕ_j^k is Killig by the definition and it is known ([3]) that the vector field ξ is Killing. Hence we have

$$\nabla_i\phi_{jk} + \nabla_j\phi_{ik} = 0, \tag{4.1}$$

(4.2) $$\nabla_i\eta_j + \nabla_j\eta_i = 0.$$

Transvecting (4.2) with η^i, we have

(4.3) $$(\nabla_t\xi_j)\eta^t = 0.$$

Transvecting (1.5) with η^e and using (4.3), we have

$$\eta^t\nabla_t\phi_{ih} = 0,$$

which and (4.1) imply

(4.4) $$\eta^t\nabla_i\phi_{ht} = 0.$$

Since $\phi_{ht}\eta^t = 0$, we find

$$(\nabla_i\phi_{ht})\eta^t + \phi_{ht}\nabla_i\eta^t = 0,$$

which and (4.4) imply

(4.5) $$(\nabla_i\eta_t)\phi_h^t = 0.$$

Substituting (4.5) into (1.5), we have

(4.6) $$\nabla_e\phi_{ih} = 0.$$

Transvecting (4.5) with ϕ_j^h, we have

$$\nabla_i\eta_j = 0,$$

which and (4.6) show that M is a cosymplectic manifold.

Thus we have

Theorem 4. *Every 3-dimensional nearly cosymplectic manifold is a cosymplectic manifold.*

5. 3-dimensional quasi cosymplectic manifolds

Let M be a 3-dimensional quasi cosymplectic manifold. Then we have

(5.1) $$\nabla_k\phi_{ji} + \phi_k^t\phi_j^s\nabla_t\phi_{si} - \eta_j\phi_k^t\nabla_t\eta_i = 0.$$

Transvecting (5.1) with η^j, we obtain

(5.2) $$\phi_k^t\nabla_t\eta_i = \phi_i^t\nabla_k\eta_t.$$

Transvecting (5.2) with $\eta^k \phi_j^i$, we find

$$\eta^s \nabla_s \eta_j = 0. \tag{5.3}$$

Transvecting (5.2) with ϕ_j^k and using (5.3), we have

$$\phi_j^s \phi_i^t \nabla_s \eta_t = -\nabla_j \eta_i. \tag{5.4}$$

Transvecting (5.1) with η^k, we have

$$\eta^k \nabla_k \phi_{ji} = 0. \tag{5.5}$$

From (1.3) and (5.4), we have

$$(\Upsilon_{ji}\Upsilon_{st} - \Upsilon_{si}\Upsilon_{jt})\nabla^s \eta^t = -\nabla_j \eta_i,$$

which and (5.3) imply

$$\Upsilon_{ji}\nabla_t \eta^t = \nabla_i \eta_j - \nabla_j \eta_i. \tag{5.6}$$

Transvecting (5.6) with g^{ji}, we obtain

$$\nabla_t \eta^t = 0. \tag{5.7}$$

Substituting (5.7) into (5.6), we find

$$\nabla_i \eta_j - \nabla_j \eta_i = 0, \tag{5.8}$$

which shows that η is closed.

From (1.5), we have

$$\phi_k^t \phi_j^s \nabla_t \phi_{si} = \phi_k^t (\nabla_t \eta_j)\eta_i. \tag{5.9}$$

From (5.1), (5.2), (5.8) and (5.9), we find

$$\nabla_k \phi_{ji} + \phi_j^t (\nabla_k \eta_t)\eta_i - \phi_k^t (\nabla_t \eta_i)\eta_j = 0. \tag{5.10}$$

By the cyclic sum of (5.10) with respect to the indices k, j and i, we find

$$\nabla_k \phi_{ji} + \nabla_j \phi_{ik} + \nabla_i \phi_{kj} = 0,$$

which means that the 2-form ϕ_{ji} is closed.

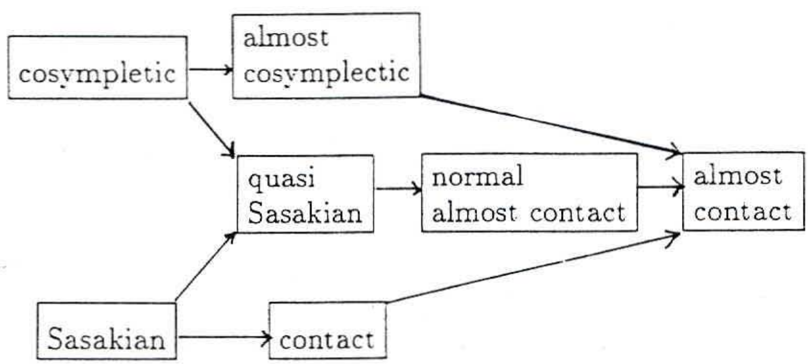
Thus we have the following

Theorem 5. *Every 3-dimensional quasi cosymplectic manifold is an almost cosymplectic manifold.*

Remark 1. In [8], Z.Olszak constructed almost cosymplectic structures with non-parallel vector field ξ on certain Lie groups in every odd dimension. Hence a 3-dimensional almost cosymplectic manifold is not cosymplectic in general.

Remark 2. Let M be a 2-dimensional manifold (surface) which does not have costant curvature 1 and let TM its tangent bundle with the fibre coordinates v^1, v^2. Then the tangent sphere bundle $\pi : T_1M \rightarrow M$ is a hypersurface of TM given by $(v^1)^2 + (v^2)^2 = 1$ and we can find a contact metric structure on T_1M which is not Sasakian. (See. Blair [1])

Remark 3. By theorems 2,3,4 and 5, the array of structures in 3-dimensional almost contact metric manifolds is reduced to the following diagram.



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