Kyungpook Mathematical Journal
Vol.34, No.2, 275-281, December 1994

Rc-Lindelöf sets and almost rc-Lindelöf sets

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In this paper we introduce the class of rc-Lindelöf sets and the class of almost rc-Lindelöf sets which represent a generalization of the well known class of S-sets. We study some basic properties of (almost) rc-Lindelöf sets and investigate the relationship between (almost) rc-Lindelöf sets and (almost) rc-Lindelöf subspaces.

1. Introduction

Recently there has been considerable interest in studying covering properties of topological spaces involving regular closed sets. In [17] Thompson introduced the class of S-closed spaces which has been extensively studied so far (see e.g.[1], [11] and [13]). A space (X,τ) is called S-closed if every regular closed cover of X has a finite subcover. If we replace in this definition "every regular closed cover" by "every countable regular closed cover" we obtain the important class of countably S-closed spaces (also known as countably rc-compact spaces [12]) which were introduced and investigated in [5]). Furthermore, Jankovic and Konstadilaki [12] defined and explored the class of rc-Lindelöf spaces, i.e. spaces in which every regular closed cover contains a countable subcover. Clearly, a space is S-closed if and only if it is countably S-closed and rc-Lindelöf. Finally, in [6] Dlaska and Ganster introduced and studied the class of almost rc-Lindelöf spaces which generalizes the class of rc-Lindelöf spaces in a natural way. A space (X,τ) is said to be almost rc-Lindelöf if every regular closed cover of X admits a countable subfamily the union of whose members is dense in (X,τ).

Closely related to the notion of an S-closed space is the concept of an S-set [3]. A subset A of a space (X,τ) is called S-set if every cover of

Received June 21, 1993.

A by regular closed sets in (X, τ) contains a finite subcover. S-sets have turned out to be useful in a variety of instances (see e.g. [2], [4], [9] and [15]).

In the present note the analogous concepts of rc-Lindelöf sets and almost rc-Lindelöf sets are introduced and investigated. We study some basic properties of (almost) rc-Lindelöf sets and explore the relationship between (almost) rc-Lindelöf sets and (almost) rc-Lindelöf subspaces.

For a subset A of a topological space (X, τ) we denote the closure of A and the interior of A by cl A and int A, respectively. The subspace topology on A is denoted by $\tau|A$. A subset G of (X, τ) is called regular open if $G = \text{int}\,(\text{cl}\,G)$. $F \subseteq X$ is said to be regular closed if $X - F$ is regular open, or equivalently, if $F = \text{cl}\,(\text{int}\,F)$. The families of regular open subsets in (X, τ) and regular closed subsets in (X, τ) are denoted by $RO(X, \tau)$ and $RC(X, \tau)$, respectively. $RO(X, \tau)$ is a base for a coarser topology τ_s on X, called the semi-regularization topology on X. A subset G is called locally dense in (X, τ) if G is dense in an open set, or equivalently, if $G \subseteq$ int (cl G). In [14] Levine called a subset $V \subseteq X$ semiopen if there exists an open set $U \subseteq X$ such that $U \subseteq V \subseteq \text{cl}\,U$. We will denote the set of natural numbers by ω and $\beta\omega$ is the Stone-Cech Compactification of ω.

2. Rc-Lindelöf sets and almost rc-Lindelöf sets

Definition. A subset A of a space (X, τ) is called an rc-Lindelöf set in (X, τ) (an almost rc-Lindelöf set in (X, τ), respectively) if for every cover of A by regular closed sets in (X, τ) there exists a countable subfamily that covers A (there exists a countable subfamily the closure of the union of whose members contains A, respectively). Since every regular closed set is semiopen and the closure of a semiopen set is regular closed the following characterization of (almost) rc-Lindelöf sets is obvious.

Lemma 2.1. *$A \subseteq X$ is an (almost) rc-Lindelöf set in (X, τ) if and only if every cover $\{V_i : i \in I\}$ of A by semiopen sets in (X, τ) admits a countable subset I_0 of I such that $A \subseteq \cup\{cl\ V_i : i \in I_0\}$ (such that $A \subseteq cl\ (\cup\{V_i : i \in I_0\})$).*

Due to the definitions, (X, τ) is an (almost) rc-Lindelöf space if and only if X is an (almost) rc-Lindelöf set in (X, τ). Moreover, it is clear that every S-set is an rc-Lindelöf set, and every rc-Lindelöf set is an almost rc-Lindelöf set. The converses, however, are not true as the following example shows.

Example 2.2. Let R be the real line with the Euclidean topology. The set Q of all rationals is clearly an rc-Lindelöf set, but it is not an S-set. Furthermore, R is almost rc-Lindelöf [6], but it is not rc-Lindelöf [7].

Lemma 2.3. *$A \subseteq X$ is an (almost) rc-Lindelöf set in (X, τ) if and only if A is an (almost) rc-Lindelöf set in (X, τ_s).*

Proof. This follows from the facts that $RC(X, \tau) = RC(X, \tau_s)$ and that for every union of regular closed sets the τ-closure coincides with the τ_s-closure.

Our next example exhibits that if $A \subseteq X$ is an (almost) rc-Lindelöf set in (X, τ) then the subspace $(A, \tau|A)$ is not necessarily an (almost) rc-Lindelöf space.

Example 2.4. Since $\beta\omega$ is S-closed [17] one easily checks that $\beta\omega - \omega$ is an S-set and thus both an rc-Lindelöf set and an almost rc-Lindelöf set in $\beta\omega$. But $\beta\omega - \omega$ is neither an rc-Lindelöf space [12] nor an almost rc-Lindelöf space [6].

In order to prove Theorem 2.6 and Theorem 2.8 the following Lemma will be very useful.

Lemma 2.5. *If A is a locally dense set in (X, τ) then*

$$RC(A, \tau|A) = \{F \cap A : F \in RC(X, \tau)\}.$$

Theorem 2.6. *Let $A \subseteq X$ be locally dense in (X, τ). Then A is an (almost) rc-Lindelöf set in (X, τ) if and only if $(A, \tau|A)$ is an (almost) rc-Lindelöf subspace.*

Proof. $(\Rightarrow)$: Assume that A is an rc-Lindelöf set in (X, τ) and let $\{F_i : i \in I\}$ be a cover of A by regular closed sets in $(A, \tau|A)$. Then, by Lemma 2.5, for every $i \in I$ there exists $B_i \in RC(X, \tau)$ such that $F_i = B_i \cap A$. Since $\{B_i : i \in I\}$ is a cover of A there exists a countable subset I_0 of I such that $A \subseteq \cup\{B_i : i \in I_0\}$. Consequently, we have $A = \cup\{F_i : i \in I_0\}$ which proves that $(A, \tau|A)$ is an rc-Lindelöf subspace. In the case that A is an almost rc-Lindelöf set in (X, τ) the proof is analogous.

$(\Leftarrow)$: Let $(A, \tau|A)$ be an rc-Lindelöf subspace and let $\{F_i : i \in I\}$ be a cover of A by regular closed sets in (X, τ). By Lemma 2.5, $\{F_i \cap A : i \in I\}$ is a cover of A by regular closed sets in $(A, \tau|A)$ and thus there exists a countable subset I_0 of I such that $A = \cup\{F_i \cap A : i \in I_0\} \subseteq \cup\{F_i : i \in I_0\}$. This shows that A is an rc-Lindelöf set in (X, τ). If we assume $(A, \tau|A)$ to be an almost rc-Lindelöf subspace then the proof is similar.

Corollary 2.7. *If $(A, \tau_s|A)$ is an open (almost) rc-Lindelöf subspace in (X, τ_s) then $(A, \tau_s|A)$ is an open (almost) rc-Lindelöf subspace in (X, τ).*

Theorem 2.8. *Let A and B be subsets of a space (X, τ) such that $A \subseteq B \subseteq X$ and B is locally dense. Then A is an (almost) rc-Lindelöf set in $(B, \tau|B)$ if and only if A is an (almost) rc-Lindelöf set in (X, τ).*

Proof. We only consider the case that A is an rc-Lindelöf set. If A is an almost rc-Lindelöf set then the proof is similar.

$(\Rightarrow)$: If $\{F_i : i \in I\}$ is a cover of A by regular closed sets in (X, τ) then A is clearly contained in the union of $\{F_i \cap B : i \in I\}$ and, by Lemma 2.5, $\{F_i \cap B : i \in I\} \subseteq RC(B, \tau|B)$. Thus there exists a countable subset I_0 of I such that $A \subseteq \cup\{F_i \cap B : i \in I_0\}$ and hence $A \subseteq \cup\{F_i : i \in I_0\}$. This shows that A is an rc-Lindelöf set in (X, τ).

$(\Leftarrow)$: Let $\{F_i : i \in I\} \subseteq RC(B, \tau|B)$ be a cover of A. By Lemma 2.5, for every $i \in I$ there exists $G_i \in RC(X, \tau)$ such that $F_i = G_i \cap B$. Obviously, $\{G_i : i \in I\}$ is a cover of A and thus we can find countably many G_i's covering A. Consequently, A is contained in the union of countably many F_i's.

Corollary 2.9. *Let A and B be open sets of a space (X, τ) such that $A \subseteq B$. Then $(A, \tau|A)$ is an (almost) rc-Lindelöf subspace of $(B, \tau|B)$ if and only if $(A, \tau|A)$ is an (almost) rc-Lindelöf subspace of (X, τ).*

Theorem 2.10. *If A is an (almost) rc-Lindelöf set in (X, τ) and $B \in RO(X, \tau)$ then $A \cap B$ is an (almost) rc-Lindelöf set in (X, τ).*

Proof. Suppose that A is an rc-Lindelöf set in (X, τ) and let $A \cap B \subseteq \cup\{F_i : i \in I\}$ where $F_i \in RC(X, \tau)$ for every $i \in I$. Then A is contained in $F = \cup\{F_i : i \in I\} \cup (X - B)$ and since $(X - B) \in RC(X, \tau)$ F is a regular closed cover of A. Thus there exists a countable subset I_0 of I such that $A \subseteq \cup\{F_i : i \in I_0\} \cup (X - B)$ and consequently $A \cap B \subseteq \cup\{F_i : i \in I_0\}$. In the case that A is an almost rc-Lindelöf set in (X, τ) the proof is analogous.

Recall that a subset $A \subseteq X$ is called θ-semiclosed [13] (θ-semiopen, respectively) if A is the intersection of regular open sets (the union of regular closed sets, respectively).

Remark 2.11. The proof of Theorem 2.10 reveals also that if A is an rc-Lindelöf set in (X, τ) and B is θ-semiclosed then $A \cap B$ is an rc-Lindelöf set in (X, τ).

Corollary 2.12. i) *If (X, τ) is an (almost) rc-Lindelöf space then every regular open set is an (almost) rc-Lindelöf subspace;*

ii) *If (X, τ) is rc-Lindelöf then every θ-semiclosed subset is an rc-Lindelöf set;*

iii) *Let A be an (almost) rc-Lindelöf set in (X, τ) and $B \in RO(X, \tau)$. Then*

1) *$A \cap B$ is an (almost) rc-Lindelöf set in $(B, \tau|B)$;*

2) *B is an (almost) rc-Lindelöf set in (X, τ) if $B \subseteq A$;*

3) *int A is an (almost) rc-Lindelöf set in (X, τ) if A is closed;*

iv) *If $(A, \tau|A)$ is an open (almost) rc-Lindelöf subspace and $B \in RO(X, \tau)$ then $(A \cap B, \tau|_{A\cap B})$ is an (almost) rc-Lindelöf subspace.*

Proof. i) follows from Theorem 2.6 and Theorem 2.10. To prove ii) apply Remark 2.11 and let $A = X$. iii) is a consequence of Theorem 2.8 and Theorem 2.10 and iv) follows from Theorem 2.6 and Theorem 2.10.

The next result is easily proved.

Proposition 2.13. *The union of countably many (almost) rc-Lindelöf sets is an (almost) rc-Lindelöf set.*

Theorem 2.14. *If A is an almost rc-Lindelöf set in (X, τ) then cl A and int(cl$A$) are almost rc-Lindelöf sets in (X, τ).*

Proof. If $F = \{F_i : i \in I\}$ is a regular closed cover of cl A then F is also a cover of A. Therefore there exists a countable subfamily I_0 of I such that $A \subseteq cl(\cup\{F_i : i \in I_0\})$ and thus cl $A \subseteq cl(\cup\{F_i : i \in I_0\})$. Since int (cl A) is regular open, by Theorem 2.10, int (cl A) is an almost rc-Lindelöf set in (X, τ).

Remark 2.15. If A is an rc-Lindelöf set then cl A and int(cl A) are not necessarily rc-Lindelöf sets. For instance, consider the real line R endowed with the Euclidean topology. Then the set Q of all rationals is an rc-Lindelöf set in R, but cl Q and int (cl Q) are not rc-Lindelöf sets since R is not rc-Lindelöf [7].

As an immediate consequence of Theorem 2.6 and Theorem 2.14 we have

Corollary 2.16. *If A is an almost rc-Lindelöf set and $B = int(cl\ A)$ then $(B, \tau|B)$ is an almost rc-Lindelöf subspace.*

To prove the next result note that if A is a semiopen subset in (X, τ) and $B \subseteq A$ is semiopen in $(A, \tau|A)$ then B is semiopen in (X, τ).

Theorem 2.17. *Let A be a subset of space (X, τ) and let $G = cl\ A$. If A is an open almost rc-Lindelöf set in (X, τ) then $(G, \tau|G)$ is an almost rc-Lindelöf subspace.*

Proof. Let $G = \cup\{V_i : i \in I\}$ where every V_i is semiopen in $(G, \tau|G)$. Since G is regular closed and thus semiopen in (X, τ), every V_i is semiopen in (X, τ). By Lemma 2.1, there exists a countable subfamily I_0 of I such that $A \subseteq cl(\cup\{V_i : i \in I_0\})$ and consequently we have $G = cl_G(\cup\{V_i : i \in I_0\})$. This shows that $(G, \tau|G)$ is an almost rc-Lindelöf subspace.

Remark 2.18. Note that Theorem 2.17 is not true if we replace "almost rc-Lindelöf" by "rc-Lindelöf". For example, consider Isbell's Ψ [10]. Then ω is an open, dense rc-Lindelöf set in Ψ, but cl ω is not rc-Lindelöf since Ψ is not rc-Lindelöf [6].

Recall that a space is said to satisfy the finite (countable) chain condition, abbreviated FCC (CCC), if every family of nonempty, pairwise disjoint open sets is finite (countable). In [4] it has been shown that a space satisfies FCC if and only if every subset is an S-set. The next result points out that spaces satisfying CCC may be characterized in terms of almost rc-Lindelöf sets.

Theorem 2.19. *The following are equivalent:*

i) *A space (X, τ) satisfies CCC;*

ii) *Every open subset is an almost rc-Lindelöf set in (X, τ);*

iii) *Every dense subset is an almost rc-Lindelöf set in (X, τ).*

Proof. Apply Proposition 2.8 in [6] and Theorem 2.6 and the result follows immediately.

In concluding this note recall that Ganster [8] has defined a space to be strongly s-regular if every open set is the union of regular closed sets (i.e. every open set is θ-semiopen). Furthermore, a space (X, τ) is called almost Lindelöf [16] if every open cover of X admits a countable subfamily the union of whose members is dense in (X, τ). The following observation is easily proved.

Proposition 2.20. *If a space is strongly s-regular then every rc-Lindelöf set is a Lindelöf subspace and every almost rc-Lindelöf set is an almost Lindelöf subspace.*

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