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ESTIMATION OF ROOTS OF PERTURBED POLYNOMIALS BY NEWTON'S INTERPOLATION FORMULA

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In this paper, we will derive some results on the perturbation of roots using Newton's interpolation formula. In addition to obtaining some new results in this area, we give a new proof of an important classical theorem of Cauchy on bounding the roots of a polynomial. And using well known absolute root bound functionals, we also compare our results with those obtained by Ostrowski [9] by giving some numerical experiments with Wilkinson's polynomials.

1. Introduction

In the practical work of solving polynomial equations, the values of the coefficients are usually rounded off, i.e., replaced by approximate values. Gautschi [2] approached the problem by deriving a condition number to measure the sensitivity of the roots with respect to small perturbation in coefficients of polynomials. Another approach is to consider the round off of the coefficients as a continuity problem and to use the geometry of the complex plane. In our work we will approach the problem on the perturbation of roots with the aid of Newton's interpolation formula and Rouche's Theorem. For more references on this approach, see [10, 13]. And then we will apply the obtained results to the following problem;

Find disks at given points containing all roots of a polynomial. Finally we give some numerical experiments with Wilkinson's polynomials for the results.

2. Definitions and some known results

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The set of all complex numbers is denoted by $\mathbf{C}$. By $B(z_0, \rho)$ we shall always mean the closed disk of radius ρ centered at z_0. If S is any bounded set in $\mathbf{C}$, its diameter is given by

$$dia(S) = SUP_{z,z' \in S}(|z - z'|).$$

Some known results from the theory of the divided differences can be found in Milne-Thomson [7].

Definition 2.1. Let $p(z)$ be a polynomial in the complex variable z. The first divided difference of $p(z)$ is denoted by $p[z_0, z_1]$ and defined by the relation

$$p[z_0, z_1] = \frac{p(z_0) - p(z_1)}{z_0 - z_1}.$$

The n-th divided difference is defined by induction in terms of the $(n-1)$-th one by the formula

$$p[z_0, \cdots, z_n] = \frac{p[z_0, \cdots, z_{n-2}, z_n] - p[z_0, \cdot, z_{n-2}, z_{n-1}]}{z_n - z_{n-1}}. \tag{2.1}$$

Lemma 2.2 [7].

$$p[z_0, \cdots, z_n] = \frac{1}{2\pi i} \int_\Gamma \frac{p(z)}{(z - z_0) \cdots (z - z_n)} dz,$$

where the points $z_0, \cdots, z_n$ lie inside the contour Γ.

By Cauchy's integral formula, we have the following estimate;

$$|p[z_0, \cdots, z_n]| \leq \frac{1}{n!} SUP_{z \in D}(|p^{(n)}(z)|), \tag{2.2}$$

where D is any convex region in the complex plane, containing $z_0, \cdots, z_n$. For $n+1$ coincident arguments z_0, we obtain the equality

$$p[z_0, \cdots, z_0] = \frac{1}{n!} p^{(n)}(z_0). \tag{2.3}$$

If $p(z)$ is a polynomial of degree n, then by Newton's interpolation formula, $p(z)$ can be reconstructed uniquely from the values of the divided differences at $z_0, \cdots, z_n$ as follows:

$$p(z) = p[z_0] + p[z_0, z_1](z - z_0) + \cdots + p[z_0, \cdots, z_n](z - z_0) \cdots (z - z_{n-1}).$$

For more information and references to these discoveries, see [7].

For $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$ with roots $q_1, \cdots, q_n$, $u(p)$ is defined by

$$u(p) = MAX(|q_i|).$$

K will denote the class of monic complex polynomials of degree n, i.e.,

$$K = \{p(z) : p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0, \text{ each } b_i \in \mathbf{C}\}.$$

A root-bound functional (rbf) on K will be a real functional M such that $M(p) \geq u(p)$ for all $p(z) \in K$.

Definition 2.3. A rbf M on K such that $M(p) = M(\tilde{p})$ whenever $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$, $\tilde{p}(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$ with $|c_i| = |b_i|$, $0 \leq i \leq n-1$, is called an absolute rbf on K.

Remark 2.4 [3, 6, 14]. By Cauchy's Theorem the unique positive root z_0 of the equation;

$$z^n - |b_{n-1}|z^{n-1} - \cdots - |b_0| = 0, \tag{2.4}$$

is an absolute root bound functional and clearly $MAX_{1 \leq i \leq n}[\sqrt[i]{\frac{|b_{n-i}|}{\alpha_i}}]$ is an optimal absolute rbf on K and equal to z_0 if we choose $\alpha_i = \frac{|b_{n-i}|}{z_0^i|}$.

For any $p(z), (\neq z^n) \in K$, we denote the corresponding z_0 as $B(p) = z_0$ and also define $B(z^n) = 0$, then B is the best absolute rbf of all absolute rbfs. While B is optimal, the positive root z_0 of the equation (2.4) can't be easily calculated. Therefore, other more computable absolute rbfs are widely used. Next we give examples of absolute rbfs which are well-known from the literature.

Let $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$.

1) $Q(P) = \begin{cases} \sum_{i=0}^{n-1} |b_i| & \text{if } \geq 1 \\ \sqrt[n]{\sum_{i=0}^{n-1} |b_i|} & \text{if } \leq 1. \end{cases}$

2) $R(P) = MIN(R_1(p), R_2(p))$, where $R_1(p) = MAX(1, \sum_{i=0}^{n-1} |b_i|)$, $R_2(p) = MAX(1 + |b_1|, \cdots, 1 + |b_{n-1}|, |b_0|)$.

3) $S(p) = 2MAX\{|b_{n-1}|, \sqrt{|b_{n-2}|}, \cdots, \sqrt[n-1]{|b_1|}, \sqrt[n]{\frac{|b_0|}{2}}\}$.

4) $T(p) = MAX\{2|b_{n-1}|, 2\frac{|b_{n-2}|}{|b_{n-1}|}, \cdots, 2\frac{|b_1|}{|b_2|}, \frac{|b_0|}{|b_1|}\}$.

Remark 2.5. Van der Sluis [14] showed that for the absolute rbf S, $S(P) \leq 2B(p)$ for all $p(z) \in K$ and hence S is nearly optimal among all absolute rbfs.

We derived the following absolute rbfs $H(p)$ and $L(p)$ in [10, 11] that in the sense of the maximum overestimation factor, are better than the nearly optimal $S(p)$ and that in particular, $H(p)$ is nearly equal to the best absolute rbf $B(p)$, see [10, 11] for more detail.

Lemma 2.6.

$$\begin{aligned} H(p) &= \frac{1}{\ln 2} MAX_{1 \leq j \leq n}(\sqrt[j]{j!|b_{n-j}|}) \\ L(p) &= \frac{1}{\ln 2} MAX\{|b_{n-1}|, \sqrt{2|b_{n-2}|}, 2\sqrt[3]{|b_{n-3}|}, \cdots, 2\sqrt[n]{|b_0|}\}. \end{aligned}$$

3. Estimation of Perturbed roots

We first introduce the following result which is closely related to our research on the perturbation of roots. The proof is found in page 221 of Ostrowski [9].

Theorem 3.1 [9]. *Let $p(z) = z^n + b_{n-1}z^{n-1} + \cdots b_0$ with roots $q_1, \cdots, q_n$, $\tilde{p}(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_0$ with roots $\tilde{q}_1, \cdots, \tilde{q}_n$.*

$$\text{(3.1)} \qquad \textit{Let } \rho(n) = \sqrt[n]{\sum_{i=1}^{n-1} |b_i - c_i| m^i}, \ \textit{where } m = MAX(|q_i|, |\tilde{q}_i|).$$

Then the roots q_i and $\tilde{q}_i$ can be ordered in such a way that

$$|q_i - \tilde{q}_i| \leq dia(c_i) - \rho(n), \quad i = 1, \cdots, n,$$

where c_i is the connected component containing q_i of $\cup_{i=1}^{n} B(q_i, \rho(n))$, and $\tilde{q}_i$ is the perturbed root of q_i for each i.

Theorem 3.2. *Let $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_0$ with roots $q_1, \cdots, q_n$, $r(z)$ some polynomial of degree $\leq n-1$, $\tilde{p}(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_0$ with roots $\tilde{q}, \cdots, \tilde{q}_n$. If $k(n)$ is the positive solution of the equation;*

$$\text{(3.2)} \qquad k^n - |r[q_1]| - |r[q_1, q_2]|k - \cdots - |r[q_1, \cdots, q_n]|k^{n-1} = 0$$

then

(1) $p(z)$ and $\tilde{p}(z)$ have the seme number of roots, counting multiplicities, in each connected component of the region $G = \cup_{i=1}^{n} B(q_i, k(n))$.

(2) If C_i is the connected component of G containing q_i, then

$$|q_i - \tilde{q}_i| \leq dia(c_i) - k(n), \quad i = 1, \cdots, n, \tag{3.3}$$

where $\tilde{q}_i$ is the perturbed root of q_i for each i.

Proof. By Descarte's rule of signs [3,12] we know that the equation (3.2) has only one positive solution. Suppose $k(n)$ is the positive solution of the equation (3.2). For $\delta > 0$, let us set $G_\delta = \cup_{i=1}^{n} B(q_i; k(n) + \delta)$. For all z on the boundary of G_δ we have

$$|z - q_i| \geq k(n) + \delta > k(n) \text{ for all } i.$$

Newton's interpolation formula gives

$$r(z) = r[q_1] + r[q_1, q_2](z - q_1) + \cdots + r[q_1, \cdots, q_n](z - q_1) \cdots (z - q_{n-1}).$$

By using the fact that

$$\frac{|r[q_1]| + |r[q_1, q_2]||z - q_1| + \cdots + |r[q_1, \cdots, q_n]||z - q_1| \cdots |z - q_{n-1}|}{|z - q_1| \cdots |z - q_n|}$$

is a decreasing function of $|z - q_i| > 0$, we have, for all z on the boundary of G_δ,

$$\begin{aligned} & \frac{|r(z)|}{|p(z)|} \\ \leq\ & \frac{|r[q_1]| + |r[q_1, q_2]|(k(n) + \delta) + \cdots + |r[q_1, \cdots, q_n]|(k(n) + \delta)^{n-1}}{(k(n) + \delta)^n} \\ <\ & \frac{|r[q_1]| + |r[q_1, q_2]|k(n) + \cdots + |r[q_1, \cdots, q_n]|k(n)^{n-1}}{k(n)^n} = 1. \end{aligned}$$

From Rouche's Theorem, we see that $p(z)$ and $\tilde{p}(z)$ have the same number of roots in each connected component of $\dot{G}_\delta$, where $\dot{G}_\delta$ is the interior of G_δ. Since δ is arbitrary, we can see that $p(z)$ and $\tilde{p}(z)$ have the same number of roots, counting multiplicities, in each connected component of G. Moreover, from the well known fact that the roots of a polynomial are continuous functions of the coefficients, the connected component containing q_i has also its perturbed root $\tilde{q}_i$. This proves the first statement (1).

For the second part (2), if c_i is the connected component containing q_i, then from the above proof the perturbed root $\tilde{q}_i$ of q_i lies in c_i. So we see that $|q_i - \tilde{q}_i| \leq dia(c_i) - k(n)$ because c_i is the union of some closed disks of radius $k(n)$.

As a corollary of Theorem 3.2, we will derive Cauchy's Theorem on bounding the roots of polynomial.

Theorem 3.3. *Let $\tilde{p}(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_0$. If $k(n)$ is the positive solution of the equation;*

$$k^n = |b_0| + |b_1|k + \cdots + |b_{n-1}|k^{n-1}, \tag{3.4}$$

then all roots of $\tilde{p}(z)$ lie in the disk $B(0, k(n))$.

Proof. Suppose that $k(n)$ is the positive solution of the equation (3.4). From Theorem 3.2, let us set $p(z) = z^n$ and $r(z) = b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$. Since each coefficient $b_i = r[0, \cdots, 0]$, the number of 0 in the bracket is $i+1$, by (1) of Theorem 3.2 we can see that $B(0, k(n))$ contains all roots of $\tilde{p}(z)$.

Remark 3.4. (1) Let q_n be the largest absolute value of the roots of $p(z)$. Then the estimate $\rho(n)$ in Theorem 3.1 increases as $|q_n|$ increases. However, the positive solution $k(n)$ of the equation (3.2) does not change when $|q_n|$ does increase because $r[q_1, q_2, \cdots, q_{n-1}, q_n] = r[q_1, q_2, \cdots, q_{n-1}, q_n']$ for any q_n'.

(2) Theorem 3.1 says that because of the $1/n$ exponent, the bounds between q_i and $\tilde{q}_i$ are weak in general for small perturbing polynomial $r(z)$.

(3) If $k(n)$ is the positive solution of the equation (3.2), then from the following example, we can see that the estimate $|q_i - \tilde{q}_i| \leq k(n)$ does not hold in general.

Consider $p(z) = z(z-0.01)(z+1.8)$ with roots $q_1 = 0, q_2 = 0.01, q_3 = -1.8$.

Now take $r(z) = -1$, then $p(z) + r(z) = z(z-0.01)(z+1.8) - 1$ with roots $\tilde{q}_1 \approx 0.6446$, $\tilde{q}_2 \approx -1.2174 + 0.2637i$, $\tilde{q}_3 \approx -1.2174 - 0.2637i$.

From the equation (3.2), we get the positive solution $k(3) = 1$. Therefore it is impossible to get $|q_i - \tilde{q}_i| \leq 1$ because $|q_2 - \tilde{q}_2| = 1.2554 > 1$.

Application of Theorem 3.2

If $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$ is given, then for any given t-points $a_1, \cdots, a_i$ in $\mathbf{C}$, we can find disks $B(a_i, \rho)$ such that the union of all disks $B(a_i, \rho)$ contains all roots of $p(z)$.

Let us consider $\tilde{p}(z) = z^4 - 12z^3 + 40z^2 - 20z + 24$ with roots

$$\begin{array}{ll} 0.1710 + 0.8062i, & 0.1710 + 0.8062i, \\ 5.8290 + 1.1649i, & 5.8290 - 1.1649i. \end{array}$$

When 0 and 5 are given, we want to find $k(4)$ so that $B(0, k(4)) \cup B(5, k(4))$ contains all roots of $\tilde{p}(z)$. To apply Theorem 3.2, choose $p(z) = z^2(z-5)^2 = z^4 - 10z^3 + 25z^2$, then

$$r(z) = \tilde{p}(z) - p(z) = -2z^3 + 15z^2 - 20z - 24.$$

From

$$r[0] = 24, r[0,0] = 20, r[0,0,5] = -5, r[0,0,5,5] = 2,$$

we get the following equation by (3.2);

$$k^4 - 2k^3 - 5k^2 - 20k - 24 = 0,$$

which has the positive solution

$$k(4) \approx 4.43, \quad (\tilde{k}(4) \approx 5.43 \text{ by rbf } (S(p)).$$

Therefore, all roots of $\tilde{p}(z)$ lie in $B(0, 4.43) \cup B(5, 4.43)$.

As background on this problem, we may consider the Gershgorin disk theorem.

< Gershgorin Disk Theorem [4, 14] >

Let $A = (a_{ij})$ be a complex matrix of order n and define the absolute off-diagonal row and column sums by $r_k = \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}|$ and $c_k = \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{jk}|$, respectively. For $k = 1, \cdots, n$, set $\bar{R}_k = \{z : |z - a_{kk}| \leq r_k\}$ and $\bar{c}_k = \{z : |z - a_{kk}| \leq c_k\}$. Then

(1) If λ is any eigenvalue of A, then $\lambda \in \bar{c}_k$ for some k and $\lambda \in \bar{R}_j$ for some j.

(2) Each component of the set $\cup_{k=1}^{n} \bar{R}_k (\cup_{k=1}^{n} \bar{c}_k)$ contains as many eigenvalues of A as points a_{ii}.

To apply the Gershgorin Theorem to the above problem, we need to find a matrix A so that the characteristic polynomial of $A = (a_{ij})$, $a_{ii} \in \{a_1, \cdots, a_t\}$ for each i, corresponds to

$$p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0.$$

However, there is little hope that such a matrix can be easily calculated unless all $a_i = 0$. But, as a special case we note that $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$ is the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ -b_0 & -b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & -b_{n-1} \end{bmatrix}.$$

Using the Gerschgorin Theorem, we have the absolute rbf $R(p)$.

4. Numerical Results

In this section we give some numerical experiments with Wilkinson's polynomials for the results obtained in section 3. And using well known absolute rbfs, we also compare our results Theorem 3.2, with Theorem 3.1 obtained by Ostrowski [9].

We note that the positive solution $k(n)$ of (3.2) can't be easily calculated. However, we can find an approximation of $k(n)$ by using previously discussed root bound functionals (rbfs) of polynomials.

(4.1) Let $\tilde{k}(n)$ be a root bound of the equation (3.2) obtained by using absolute rbfs.

In order to apply (3.1), we introduce a way to find a convenient estimate of $\rho(n)$ as follows; Let $\tilde{m} = MAX\{M(p), M(\tilde{p})\}$, where M is an absolute rbf.

(4.2) $$\text{Set } \bar{\rho}(n) = \sqrt[n]{\sum_{i=0}^{n-1} |b_i - c_i| \tilde{m}^i}.$$

So we have

$$|q_i - \tilde{q}_i| \le dia(\bar{c}_i) - \bar{\rho}(n), \quad i = 1, \cdots, n,$$

where $\bar{C}_i$ is the connected component containing q_i of $\cup_{i=1}^n B(q_i, \bar{\rho}(n))$.

Let $w(z) = (z-1)(z-2)\cdots(z-n)$ and $r(z) = 0.002z^{n-1}$. And set

$$(4.3) \qquad \mathcal{B}(n) = MAX(|q_i - \tilde{q}_i|).$$

Let us recall that $k(n)$ is the positive solution of the equation;

$$(4.4) \qquad k^n - |r[q_1]| - |r[q_1,q_2]|k - \cdots - |r[q_1,\cdots,q_n]|k^{n-1} = 0.$$

And recall that

$$(4.5) \qquad \rho(n) = \sqrt[n]{\sum_{i=0}^{n-1} |b_i - c_i| m^i}, \text{ where } m = MAX(|q_i|, |\tilde{q}_i|).$$

Then we obtain the following numerical results.

Table

Numerical experiments with Wilkinson's polynomial

deg $w(z)$	$\rho(n)$	$\bar{\rho}(n)$	$k(n)$	$\bar{k}(n)$	$\mathcal{B}(n)$
6	1.580	6.090	0.787	1.180	0.278
7	2.182	9.801	1.142	1.700	0.754
8	2.837	10.58	1.572	2.555	1.100
9	3.575	14.78	2.075	3.386	1.539
10	4.449	27.52	2.655	4.195	2.071
11	5.408	35.77	3.314	5.332	2.651
12	6.453	45.23	4.044	6.569	3.279
13	7.581	55.93	4.854	7.900	3.955
14	8.793	67.90	5.740	9.325	4.680
15	10.09	81.41	6.704	10.84	5.491
20	17.76	171.3	12.67	20.67	10.33

For $r(z) = -2^{-23}z^{19}$,

20	10.92	74.34	3.961	6.894	3.221

(Note that to obtain the minimal estimates, we used the absolute rbf $S(p)$ for $\bar{k}(n)$ for $n \geq 8$, and $R(p)$ for $n = 6, 7$, and used $L(p)$ for $\bar{\rho}(n)$.)

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