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LITTLEWOOD-PALEY FUNCTIONS ON GROUPS OF HOMOGENEOUS TYPE

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1. Introduction

Let G be a group of homogeneous type which is a more general setting than R^n. Then these groups form a natural habitat for extensions of many of the objects studied in Euclidean harmonic analysis. In 1985, R.R. Coifman, Y. Meyer and E.M. Stein [1] introduced the tent spaces on the upper half-space R^{n+1}_+ which are well adapted for the study of a variety of questions related to harmonic analysis and its applications. In this paper, we will develop the theory of the tent spaces on $G \times (0, \infty)$, which is an analogue of the upper half-space R^{n+1}_+. The purpose of this paper is to show that there is a connection between the tent spaces $T^p_2(G \times (0, \infty))$ and the atomic Hardy spaces $H^p_{at}(G)$ for $0 < p \leq 1$. To do this, we shall consider the operator $\mathcal{K}(f)$ defined on $T^p_2(G \times (0, \infty))$ for $0 < p < \infty$, by

$$\mathcal{K}(f) = \int_0^\infty f(\,\cdot\,, t) * \psi_t \frac{dt}{t},$$

which ψ is an appropriate Littlewood-Paley function defined on G.

2. Terminologies and notations

Let G be a topological group endowed with a positive measure μ on G. Assume that d is a pseudo-metric on G, i.e., a nonnegative function defined on $G \times G$ satisfying

(i) $d(x, x) = 0$; $d(x, y) > 0$ if $x \neq y$,

(ii) $d(x, y) = d(y, x)$, and

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(iii) $d(x,z) \leq K[d(x,y)+d(y,z)]$, where K is some fixed constant. Assume further that

(G1) for $\rho > 0$, the balls $B(x,\rho) = \{y \in G : d(x,y) < \rho\}$ form a basis of open neighborhoods at $x \in G$, and that μ satisfies the doubling property

(G2) $0 < \mu[B(x,2\rho)] \leq A\mu[B(x,\rho)] < \infty$, where A is some fixed constant. Finally we assume that μ is left-invariant:

(G3) $\mu(xE) = \mu(E)$ for each $x \in G$ and any measurable set $E \subset G$, and

(G4) $\mu(E^{-1}) = \mu(E)$,

(G5) d is left-invariant:

Then the triple (G,d,μ) is called a *group of homogeneous type*. Let (G,d,μ) be a group of homogeneous type. Then, for $\rho > 0$, an automorphism δ_ρ of G which is called a *dilation* on G is assumed to satisfy

$$\mu[\delta_\rho(E)] = \rho^n \mu(E) \tag{2.1}$$

for some fixed positive integer n and any measurable set $E \subset G$.

It is known that d is left-invariant if and only if

$$d(x,y) = |x^{-1}y|,$$

where $|\cdot|$ is a nonnegative function on G satisfying

(i) $|x| = 0$ if and only if $x = e$,

(ii) $|xy| \leq K(|x|+|y|)$, where K is some fixed constant, and

(iii) $|x^{-1}| = |x|$.

For details see [5].

For $x, y \in G$, and $\rho > 0$, the set

$$B(x,\rho) = \{y \in G : |x^{-1}y| < \rho\}$$

is called the *ball* centered at $x \in G$ with a radius ρ. Now consider the space $G \times (0,\infty)$,which is called the *upper half-space* over G. For any $\alpha > 0$, and $x \in G$, the set

$$\Gamma_\alpha(x) = \{(y,t) \in \times(0,\infty) : |x^{-1}y| < \alpha t\}.$$

is called the *cone* of aperture α whose vertex is $x \in G$. For simplicity, we put $\Gamma(x) = \Gamma_1(x)$.

For any closed subset $F \subset G$, and any $\alpha > 0$, let $\mathcal{R}^{(\alpha)}(F) = \cup_{x \in F}\Gamma_\alpha(x)$. For simplicity, we put $\mathcal{R}(F) = \mathcal{R}^{(1)}(F)$. Let O be an open subset of G

which is the complement of F, $O = F^c$. Then the *tent* over O, denoted by $T(O)$, is given as $T(O) = \mathcal{R}(F)^c$.

For a function f defined on $G \times (0, \infty)$, we define a functional $\mathcal{A}(f)$, for $x \in G$, by

$$\mathcal{A}(f)(x) = [\int_{\Gamma(x)} |f(y,t)|^2 \frac{d\mu(y)dt}{t^{n+1}}]^{1/2}.$$

Then the *tent space* $T_2^p(G \times (0,\infty))$ is defined as the space of functions f on $G \times (0,\infty)$, so that $\mathcal{A}(f) \in L^p(d\mu)$ when $0 < p < \infty$. Define $||f||_{T_2^p} = ||\mathcal{A}(f)||_p$. The fundamental principle is that these spaces have an atomic decomposition. Here we will deal with $T_2^p(G \times (0,\infty))$ for $0 < p \leq 1$. A function $a(x,t)$ defined on $G \times (0,\infty)$ is said to be a T_2^p*-atom* if

(i) $a(x,t)$ is supported in the tent $T(B)$ for some ball B in G, and

(ii) $\left\{\frac{1}{\mu(B)} \int_{T(B)} |a(x,t)|^2 \frac{d\mu(x)dt}{t}\right\}^{1/2} \leq [\mu(B)]^{-1/p}$.

Note that the constant C will be used without any explicit explanation throughout this paper.

Lemma 1 ([8]). *There exists a constant C so that if $f \in T_2^p(G\times(0,\infty))$ for $0 < p \leq 1$, then there exist a sequence $\{a_j\}$ of T_2^p-atoms, and a sequence $\{\lambda_j\}$ of positive numbers such that*

$$|f(x,t)| \leq \sum_{j=1}^{\infty} \lambda_j a_j(x,t), \text{ and } \sum_{j=1}^{\infty} \lambda_j^p \leq C||\mathcal{A}(f)||_p^p.$$

We are now going to introduce the atomic Hardy spaces associated with a group of homogeneous type G. For $0 < p < q$, $p \leq 1 \leq q \leq \infty$, a function $a(x)$ defined on G is said to be a H_q^p*-atom* if

(i) $a(x)$ is supported in some ball B in G,

(ii) $\left\{\frac{1}{\mu(B)} \int_B |a(x)|^q d\mu(x)\right\}^{1/q} \leq [\mu(B)]^{-1/p}$ if $q < \infty$, or $||a||_\infty \leq [\mu(B)]^{-1/p}$ if $q = \infty$,

(iii) $\int_G a(x) d\mu(x) = 0$.

In the present setting we introduce an appropriate space of linear functionals in order to define the atomic Hardy spaces. In order to do this, we introduce the *Lipschitz spaces* $\mathcal{L}_\alpha(G)$, $\alpha > 0$, consisting of those functions l on G for which

$$|l(x) - l(y)| \leq C[\mu(B)]^\alpha,$$

where B is any ball containing both x and y and C depends only on l.

We now define the space $H^{p,q}_{at}(G)$ for $0 < p < 1 \leq q \leq \infty$, to be the subspace of the dual $\mathcal{L}^*_\alpha(G)$ of $\mathcal{L}_\alpha(G)$, where $\alpha = 1/p - 1$, consisting of those linear functionals h which admit an *atomic decomposition*

$$h = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where the a_j's are H^p_q-atoms, the λ_j's are positive numbers, and $\sum_{j=1}^{\infty} \lambda_j^p < \infty$. The infimum of $\sum_{j=1}^{\infty} \lambda_j^p$ taken over all such decompositions of h is denoted by $||h||_{H^{p,q}_{at}}$. Then R.R. Coifman and G. Weiss [4] show that $H^{p,q}_{at}(G) = H^{p,\infty}_{at}(G)$ whenever $0 < p < q < \infty$. This enables us to define the *atomic Hardy space* $H^p_{at}(G)$ for $0 < p \leq 1$, to be any one of the spaces $H^{p,q}_{at}(G)$ for $0 < p < q \leq \infty$, $1 \leq q$.

A function ψ defined on G is said to be a *Littlewood-Paley function* provided it satisfies the following properties:

(2.2 a) ψ has a compact support in the unit ball,

(2.2 b) $|\psi(x)| \leq C(1 + |x|)^{-(n+\alpha)}$ for some $\alpha > 0$,

(2.2 c) $\int_G |\psi(xy) - \psi(x)| d\mu(x) \leq C|y|^\gamma$ for all y in G and some $\gamma > 0$, and

(2.2 d) $\int_G \psi(x) d\mu(x) = 0$.

For $t > 0$, we define ψ_t by

$$\psi_t = t^{-n}\psi \circ \delta_{1/t}, \text{ i.e. }, \quad \psi_t(x) = t^{-n}\psi(\frac{x}{t}).$$

Let f and g be measurable functions defined on G. Then the *convolution* $f * g$ of f and g is defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)d\mu(y) = \int_G f(xy^{-1})g(y)d\mu(y)$$

for all x such that the integral exists. For ψ as in (2.2), we consider the operator $\mathcal{K}(f)$ defined on $T^p_2(G \times (0,\infty))$ for $0 < p < \infty$, by

$$\mathcal{K}(f) = \int_0^\infty f(\cdot, t) * \psi_t \frac{dt}{t}. \tag{2.3}$$

3. Main results

Lemma 2. *Suppose $1 < p < \infty$. Let $f \in T^p_2(G \times (0,\infty))$ and $g \in T^{p'}_2(G \times (0,\infty))$, with $1/p + 1/p' = 1$. Then we have*

$$\int_{G\times(0,\infty)} |f(y,t)g(y,t)| \frac{d\mu(y)dt}{t} \leq C_n^{-1} \int_G \mathcal{A}(f)(x)\mathcal{A}(g)(x)d\mu(x),$$

where C_n is the volume of the unit ball.

Proof. Suppose $1 < p < \infty$. Let $f \in T_2^p(G \times (0,\infty))$ and $g \in T_2^{p'}(G \times (0,\infty))$, with $1/p + 1/p' = 1$. Then it follows from Schwarz's inequality that

$$\begin{aligned}
&\int_{G\times(0,\infty)} |f(y,t)g(y,t)|\frac{d\mu(y)dt}{t} \\
&= C_n^{-1}\int_G [\int_{G\times(0,\infty)} \chi_{B(y,t)}(x)|f(y,t)g(y,t)|\frac{d\mu(y)dt}{t^{n+1}}]d\mu(x) \\
&= C_n^{-1}\int_G [\int_{\Gamma(x)} |f(y,t)g(y,t)|\frac{d\mu(y)dt}{t^{n+1}}]d\mu(x) \\
&= C_n^{-1}\int_G \mathcal{A}(f)(x)\mathcal{A}(g)(x)d\mu(x),
\end{aligned}$$

where C_n is the volume of the unit ball, and $\chi_{B(y,t)}$ is the characteristic function of the ball $B(y,t)$ of radius t centered at y. Thus

$$\int_{G\times(0,\infty)} |f(y,t)g(y,t)|\frac{d\mu(y)dt}{t} \le C_n^{-1}\int_G \mathcal{A}(f)(x)\mathcal{A}(g)(x)d\mu(x),$$

which completes the proof.

Theorem 3. *Let $1 < p < \infty$. Then the operator $\mathcal{K}$, defined as (2.3), can be extended to a bounded linear operator from $T_2^p(G \times (0,\infty))$ to $L^p(d\mu)$.*

Proof. Let $f \in T_2^p(G \times (0,\infty))$ for $1 < p < \infty$. Then it suffices to bound $\int_G \mathcal{K}(f)(x)g(x)d\mu(x)$ for $g \in L^{p'}(d\mu)$, where $1/p + 1/p' = 1$. Let $\tilde{g}(x,t) = g * \tilde{\psi}_t(x)$, with $\tilde{\psi}_t(x) = \psi(x^{-1}/t)t^{-n}$. Then, by Hölder's inequality, it readily follows from Lemma 2 that

$$\begin{aligned}
(3.1)\quad |\int_G \mathcal{K}(f)(x)g(x)d\mu(x)| &\le \int_{G\times(0,\infty)} |f(y,t)\tilde{g}(y,t)|\frac{d\mu(y)dt}{t} \\
&\le C_n^{-1}\int_G \mathcal{A}(f)(x)\mathcal{A}(\tilde{g})(x)d\mu(x) \\
&\le C_n^{-1}||\mathcal{A}(f)||_p||\mathcal{A}(\tilde{g})||_{p'},
\end{aligned}$$

where C_n is the volume of the unit ball. Now it is true that, with slight modifications of [6, Ch.3], we have $||\mathcal{A}(\tilde{g})||_{p'} \le C||g||_{p'}$ for some constant C. Therefore the last side of (3.1) is less than $C||f||_{T_2^p}||g||_{p'}$. Thus

$$|\int_G \mathcal{K}(f)(x)g(x)d\mu(x)| \le C||f||_{T_2^p}||g||_{p'}$$

for some constant C, and $||\mathcal{K}(f)||_p \leq C||f||_{T_2^p}$, which completes the proof.

Theorem 4. *Let $0 < p \leq 1$. Then the operator $\mathcal{K}(f)$, defined as (2.3), can be extended to a bounded linear operator from $T_2^p(G \times (0,\infty))$ to $H_{at}^p(G)$.*

Proof. Let $f \in T_2^p(G \times (0,\infty))$ for $0 < p \leq 1$. Then by Lemma 1, f can be written as

$$|f(x,t)| \leq \sum_{j=1}^{\infty} \lambda_j a_j(x,t),$$

where the a_j's are T_2^p-atoms, the λ_j's are positive numbers, and $\sum_{j=1}^{\infty} \lambda_j^p \leq C||\mathcal{A}(f)||_p^p$. Thus it suffices to show that $\mathcal{K}(a)$ maps a T_2^p-atom to a bounded multiple of an H_2^p-atom for $0 < p \leq 1$. Let $a(x,t)$ be such a T_2^p-atom, associated to the ball B. Then $a(x,t)$ is supported in $T(B)$. Since ψ is supported in the unit ball, it follows from the definition of $\mathcal{K}$ that $\mathcal{K}(a)$ is supported in the ball B^* having the same center as B, but twice the radius. Moreover, by the proof of Theorem 3,

$$\begin{aligned} \int_G |\mathcal{K}(a)|^2 d\mu(x) &\leq C||a||_{T_2^2}^2 \\ &= C\int_{T(B)} |a(y,t)|^2 \frac{d\mu(y)dt}{t} \\ &\leq C[\mu(B)]^{1-2/p}. \end{aligned}$$

Finally, $\mathcal{K}(a)$ satisfies

$$\int_G \mathcal{K}(a)(x)d\mu(x) = 0,$$

since $\int_G \psi(x)d\mu(x) = 0$. The proof is therefore completed.

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