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A NEW CLASS OF CONTIGUOUS FUNCTION RELATIONS FOR GENERALIZED HYPERGEOMETRIC FUNCTIONS BY USING A DIFFERENCE OPERATOR

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The object of this paper is to give a new class of contiguous function relations for certain generalized hypergeometric functions.

1. Introduction

Recently, we [1] introduced a class of generalized hypergeometric functions $I_{n;(b_q)}^{\alpha;(a_p)}(x,w)$ defined by using a difference operator and also derived the following relation:

$$(1.1)\qquad I_{n;(b_q)}^{\alpha;(a_p)}(x,w)=\frac{(1+\alpha)_n}{n!}F_{q:1;0}^{p:2;1}\left[\begin{matrix}(a_p):-n,\frac{x}{w};-\frac{x}{w};\\(b_q):1+\alpha;-;\end{matrix}\;w,w\right],$$

where $F_{q:s;v}^{p:r;u}(x,y)$ is a double hypergeometric function (see Srivastava and Karlsson [5, p.27(28)]).

The following notations of a difference operator given by Milne-Thomson [2], and by Parihar and Patel [3], have throughout been adopted:

$$(1.2)\qquad \Delta_{x,w}f(x)=\frac{f(x+w)-f(x)}{w}$$

$$(1.3)\qquad \Delta_{x,w}^{n}(u_x\cdot v_x)=\sum_{k=0}^{n}\binom{n}{k}\Delta_{x,w}^{n-k}u_{x+kw}\Delta_{x,w}^{k}v_x$$

$$(1.4)\qquad \Delta_{x,w}\left(\frac{x}{w}\right)_R=\left(\frac{R}{w}\right)\left(\frac{x}{w}+1\right)_{R-1}$$

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and

$$\Delta_{x,w}(-\frac{x}{w})_R = (-\frac{R}{w})(-\frac{x}{w}+1)_{R-1}. \tag{1.5}$$

In this paper, we have assumed the following notations for convenience

$$(1.6)\ I_{n;(b_q)}^{\alpha;(a_p)}(x^{*,r}, w) = \frac{(1+\alpha)_n}{n!} F_{q:1;0}^{p:2;1}\left[\begin{array}{l}(a_p): -n, \frac{x}{w}+r; -\frac{x}{w}; \\ (b_q): 1+\alpha; -;\end{array} w, w\right],$$

$$(1.7)\ I_{n;(b_q)}^{\alpha;(a_p)}(x_{*,r}, w) = \frac{(1+\alpha)_n}{n!} F_{q:1;0}^{p:2;1}\left[\begin{array}{l}(a_p): -n, \frac{x}{w}; -\frac{x}{w}+r; \\ (b_q): 1+\alpha; -;\end{array} w, w\right],$$

and

$$(1.8) I_{n;(b_q)}^{\alpha;(a_p)}({}^{*,r}x, w) = \frac{(1+\alpha)_n}{n!} F_{q:1;0}^{p:2;1}\left[\begin{array}{l}(a_p): -n, \frac{x}{w}+r; -\frac{x}{w}+r; \\ (b_q): 1+\alpha; -;\end{array} w, w\right].$$

2. The main contiguous function relations

We have derived the following relations of contiguous functions in the direction given by Rainville [4].

(a) The $(p+q-1)$ simple relation; Using the operator $\theta = x\cdot\Delta_{x,w}$, we see that

$$\begin{aligned}(2.1)\qquad & (b_q)(\theta+a_r)I_{n;(b_q)}^{\alpha;(a_p)}(x,w) \\ = \ & (b_q)(a_r)I_{n;(b_q)}^{\alpha;((a_r)+),(a_{p;r})}(x,w) - x(a_p)\{I_{n;((b_q)+)}^{\alpha;((a_p)+)}(x^{*,1},w) \\ & -I_{n;((b_q)+)}^{\alpha;((a_p)+)}(x_{*,1},w)\},\end{aligned}$$

where $r = 2,\cdots,p$. Similarly, it follows that

$$\begin{aligned}(2.2)\qquad & (b_q)(\theta+b_r-1)I_{n;(b_q)}^{\alpha;(a_p)}(x,w) \\ = \ & (b_q)(b_r-1)I_{n;(b_r)-),(b_{q;r})}^{\alpha;(a_p)}(x,w) + (a_p)x\{I_{n;((b_q)+)}^{\alpha;((a_p)+)}(x_{*,1},w) \\ & -I_{n;((b_q)+)}^{\alpha;((a_p)+)}(x^{*,1},w)\};\end{aligned}$$

where $r = 1, 2, \cdots, q$.

The equations (2.1) and (2.2) follow at once by elimination of $\theta I_{n;(b_q)}^{\alpha;(a_p)}(x,w)$ to $(p+q-1)$ linear algebraic relations between $I_{n;(b_q)}^{\alpha;(a_p)}(x,w)$ and pairs of its contiguous functions. Let us use $I_{n;(b_q)}^{\alpha;((a_1)+),(a_{p:1})}(x,w)$; as an element in each equation. We get the following results:

$$(2.3)\qquad \begin{aligned}&(a_1-a_r)I_{n;(b_q)}^{\alpha;(a_p)}(x,w)\\ =\;& a_1 I_{n;(b_q)}^{\alpha;((a_1)+),(a_{p;1})}(x,w)-a_r I_{n;(b_q)}^{\alpha;((a_r)+),(a_{p;r})}(x,w);\\ & r=2,3,\cdots,p\end{aligned}$$

and

$$(2.4)\qquad \begin{aligned}&(a_1-b_{r+1})I_{n;(b_q)}^{\alpha;(a_p)}(x,w)\\ =\;&(1-b_r)I_{n;((b_r)-),(b_{q;r})}^{\alpha;(a_p)}(x,w)+a_1 I_{n;(b_q)}^{\alpha;((a_r)+),(a_{r;1})}(x,w);\\ & r=1,2,\cdots,q.\end{aligned}$$

(b) A relation involving $(q+1)$ contiguous functions:
From (1.1), we get

$$I_{n;(b_q)}^{\alpha;(a_p)}(x,w)=\frac{(1+\alpha)_n}{n!}\sum_{k=1}^{n}\sum_{i=0}^{\infty}\frac{[(a_p)]_{k+i}(-n)_k(\frac{x}{w})_k(-\frac{x}{w})_i w^{k+i}}{[(b_q)]_{k+1}(1+\alpha)_k k!i!},$$

it follows that

$$(2.5)\qquad \begin{aligned}&\theta I_{n;(b_q)}^{\alpha;(a_p)}(x,w)\\ =\;& -\frac{x(a_p)}{(b_q)}I_{n;((b_q)+)}^{\alpha;((a_p)+)}(x^{*,1},w)-\frac{x(2+\alpha)_{n-1}}{(n-1)!}\\ &\sum_{k=0}^{n-1}\sum_{i=0}^{\infty}S_{k+i}\frac{[(a_p)]_{k+i}(-(n-1))_k(\frac{x}{w}+1)_k(-\frac{x}{w})_i w^{k+i}}{[(b_q)]_{k+i}(2+\alpha)_k k!i!},\end{aligned}$$

where $S_n=\frac{(a_1+n)(a_2+n)\cdots(a_p+n)}{(b_1+n)(b_2+n)\cdots(b_q+n)}$. Now, if $p<q$, then the degree of the numerator of S_{k+i} is lower than the degree of the denominator and the elementary theory of rational fraction expansions yields (Rainville [4, p.717])

$$S_{k+i}=\sum_{j=1}^{q}\frac{b_jU_j}{b_j+n},\quad p<q,$$

where $U_j=\frac{\prod_{s=1}^{p}(a_s-b_j)}{b_j\prod_{s=1,(j)}^{q}(b_s-b_j)}$. Therefore

$$(2.6)\qquad \begin{aligned}\theta I_{n;(b_q)}^{\alpha;(a_p)}(x,w) &= -\frac{x(a_p)}{(b_q)}I_{n;((b_q)+)}^{\alpha;((a_p)+)}(x^{*,1},w)\\ &\quad -x\sum_{j=1}^{q}U_jI_{n-1;((b_q)+),(b_{q;j})}^{\alpha+1;(a_p)}(x^{*,1},w).\end{aligned}$$

The elimination of $\theta I_{n;(b_q)}^{\alpha;(a_p)}(x,w)$, using (2.6) and the case $r=1$ of (2.1), leads to

$$\begin{aligned}(2.7)\qquad & (a_1)(b_q)I_{n;(b_q)}^{\alpha;(a_p)}(x,w)\\ =\; & (a_1)(b_q)I_{n;(b_q)}^{\alpha;((a_p)+),(a_{p;1})}(x,w)+x(a_p)I_{n;((b_q)+)}^{\alpha;((a_p)+)}(x_{*,1},w)\\ & +(b_q)x\sum_{j=1}^{q}U_jI_{n-1;((b_j)+),(b_{q;j})}^{\alpha+1;(a_p)}(x^{*,1},w).\end{aligned}$$

Similarly, we can obtain the following results.

$$\begin{aligned}(2.8)\qquad & (b_q)a_1I_{n;(b_q)}^{\alpha;(a_p)}(x,w)\\ =\; & (a_1)(b_q)I_{n;(b_q)}^{\alpha;((a_p)+),(a_{p;1})}(x,w)\\ & +x(a_p)I_{n;((b_q)+)}^{\alpha;((a_p)+)}(x_{*,1},w)+xI_{n-1;(b_q)}^{\alpha+1;(a_p)}(x^{*,1},w)\\ & +x\sum_{j=1}^{q}U_jI_{n-1;((b_j)+),(b_{q;j})}^{\alpha+1;(a_p)}(x^{*,1},w),\quad p=q,\end{aligned}$$

$$\begin{aligned}(2.9)\qquad & (b_q)a_1I_{n;(b_q)}^{\alpha;(a_p)}(x,w)\\ =\; & (a_1)(b_q)I_{n;(b_q)}^{\alpha;((a_p)+),(a_{p;1})}(x,w)\\ & +x(a_p)I_{n;((b_q)+)}^{\alpha;((a_p)+)}(x_{*,1},w)+x(A-B)I_{n-1;(b_q)}^{\alpha+1;(a_p)}(x^{*,1},w)\\ & -(a_p)x^2I_{n-1;((b_q)+)}^{\alpha+1;((a_p)+)}(*,1_x,w)-(a_p)x(x-w)I_{n-2;((b_q)+)}^{\alpha+2;((a_p)+)}(x^{*,2},w)\\ & +x\sum_{j=1}^{q}U_jI_{n-1;((b_j)+),(b_{q;j})}^{\alpha+1;(a_p)}(x^{*,1},w),\quad p=q+1.\end{aligned}$$

where $A=\sum_{s=1}^{p}a_s$ and $B=\sum_{s=1}^{q}b_s$.

(c) A relation involving p relations:

Now replacing a_r by a_{r-1} in (1.1) and using the same technique as applied in above results, we have the following relations:

$$\begin{aligned}(2.10)\qquad & (b_q)I_{n;(b_q)}^{\alpha;(a_p)}(x,w)\\ =\; & (b_q)I_{n;(b_q)}^{\alpha;((a_r)-),(a_{p;r})}(x,w)-x(a_{p;r})I_{n;((b_q)+)}^{\alpha;(a_r),((a_{p;r})+)}(x_{*,1},w)\\ & -(b_q)x\sum_{j=1}^{q}W_{j,r}I_{n-1;((b_j)+),(b_{q;j})}^{\alpha+1;(a_p)}(x^{*,1},w);\quad p\le q, r=1,2,\cdots,p,\end{aligned}$$

where $W_{j,k} = \frac{\prod_{s=1,(k)}^{p}(a_s-b_j)}{b_j\prod_{s=1,(j)}^{q}(b_s-b_j)}$ and

$$
\begin{aligned}
(2.11)\qquad & (b_q) I_{n;(b_q)}^{\alpha;(a_p)}(x,w) \\
= \; & (b_q) I_{n;(b_q)}^{\alpha;((a_r)-),(a_{p;r})}(x,w) - x(a_{p;r}) I_{n;((b_q)+)}^{\alpha;(a_r),((a_{p;r})+)}(x_{*,1},w) \\
& -x I_{n-1;(b_q)}^{\alpha+1;(a_p)}(x^{*,1},w) \\
& -x \sum_{j=1}^{q} W_{j,r} I_{n-1;((b_j)+),(b_{q;j})}^{\alpha+1;(a_p)}(x^{*,1},w); r = 1,2,\cdots,p,
\end{aligned}
$$

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References

[1] M. Lahiri and B. Satyanarayana, *A class of generalized hypergeometric functions defined by using a difference operator*, Soochow J. Math. 19(1993), 163-171.

[2] L. M. Milne-Thomson, *The Calculus of Finite Differences*, Macmillan, London, 1951.

[3] C. L. Parihar and V. C. Patel, *On modified Jacobi polynomials*, J. Indian Acad. Math. 1(1979), No.1, 41-46.

[4] E. D. Rainville, *The contiguous function relations for F_q with applications to Bateman's $J_n^{u,v}$ and Rice's $H_n(\xi^p,p,v)$*, Bull. Amer. Math. Soc. 51(1945), 714-723.

[5] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1985.

[6] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1984.

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