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ON THE BEST APPROXIMATION BY RATIONAL FUNCTIONS WITH FIXED POLES IN $H_q^p(p \geq 1, q > 1)$ SPACES

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If a function $f(z)$ is analytic in the unit disc $|z| < 1$ and satisfies the condition

$$\iint_{|z|<1} (1-|z|^2)^{q-2}|f(z)|^p d\sigma_z < +\infty$$

for parameters p and q, where $z = x + iy$, $d\sigma_z = dxdy$, we say that the function $f(z)$ belongs to H_q^p spaces. The condition $q > 1$ is needed, so that constant-functions $f(z) = C$ belong to H_q^p spaces.

Suppose

$$\|f(z)\| = [\iint_{|z|<1} (1-|z|^2)^{q-2}|f(z)|^p d\sigma_z]^{\frac{1}{p}}.$$

It is easy to prove that H_q^p spaces are Banach spaces if $p \geq 1$ and $q > 1$, but Frechet spaces if $0 < p < 1$ and $q > 1$.

In [1], Charles K. Chui and Xie-chang Shen gave a formula expressing the function in H_q^p spaces. In [2] and [3], we proved the theorems about the estimation of the order of the best approximation by polynomials in $H_q^p(p \geq 1, q > 1)$ spaces and their inverse theorems. In [4] and [5], we made a research on or studied the Hardy-Littlewood-type theorems and the best approximation by polynomials in $H_q^p(0 < p < 1, q > 1)$ spaces. In this paper, we are going to make a research on the estimation of the order of the best approximation by rational functions with fixed poles in H_q^p spaces for parameters $p \geq 1$ and $q > 1$.

Let $Z = \{z_1, z_2, \cdots, z_n\}$ be a finite sequence of (possibly repeated) points in the extended complex plane, $|z_k| > 1 (k = 1, 2, \cdots, n)$. By $R(z)$

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we denote the class of rational functions with poles among the points occuring in Z and of a maximal multiplicity equal to the number of repetitions. Hence $r(z) \in R(Z)$ if and only if

$$r(z) = P(z) \prod_{\zeta = Z} (z-\zeta)^{-1}$$

where $P(z)$ is a polynomial of degree at most n if $Z = \{z_k\}_{k=1}^{n}$. If $\zeta = \infty$, then $(z-\zeta)^{-1}$ is interpreted as 1.

Let $\rho(f, z)$ be the best approximation error of the function $f(z)$ by the functions of $R(Z)$ class in H_q^p spaces:

$$\rho(f,z) = \inf_{r(z) \in R(Z)} \{\|f(z) - r(z)\|\}.$$

Denoting

$$\lambda = \sum_{k=1}^{n} (1 - \frac{1}{|z_k|})$$

noting the result of [2], and using the methods of [6] and [7], we can obtain the following main result of this paper:

Theorem. *For any function $f(z) \in H_q^p (p \geq 1, q > 1)$ we have*

$$\rho(f, Z) \leq A_1 \int_0^{\frac{1}{\lambda}} \frac{\omega(t,f)}{t} dt + Ce^{-\frac{1}{10}\lambda},$$

where $\omega(t, f)$ if the integral modulus of the function $f(z)$ in the sense of H_q^p space:

$$\omega(t,f) = \sup_{|h| \leq t} \{\|f(z+h) - f(z)\|\},$$

and A_1 is a constant independent of λ and f, C is a constant independent of λ.

It is easy to see that the result of [7] is a particular condition of the result of this paper.

In order to prove the Theorem of this paper, we need introduce two lemmas first.

Lemma 1. *Suppose a function $Q(z)$ is analytic in the unit disc $|z_0| \leq \frac{1}{2}$ then we have the inequality*

$$|Q(z_0)|^p \leq \frac{4^s}{\pi} \|Q(z_0)\|^p$$

for $p > 0,\ q > 1,$ *in which* $s = \max\{2, q\}$.

Proof. As $Q(z)$ is analytic in the unit disc we have

$$\int_1^{2\pi} |Q(z_0)|^p d\theta \leq \int_0^{2\pi} |Q(z_0 + re^{i\theta})|^p d\theta \tag{1}$$

for $p > 0$ and $r < 1 - |z_0|$.

Multiplying two sides of (1) by rdr and integrating them from 0 to $\frac{1}{2}(1 - |z_0|)$, we have

$$\pi |Q(z_0)|^p (\frac{1 - |z_0|}{2})^2 \leq \int\int_{|z - z_0| < \frac{1}{2}(1 - |z_0|)} |Q(z)|^p d\sigma. \tag{2}$$

Then, if $q \leq 2$, we can obtain

$$\begin{aligned}
\|Q(z)\|^p &\geq \int\int_{|z-z_0|<\frac{1}{2}(1-|z_0|)} (1 - |z|^2)^{q-2} |Q(z)|^p d\sigma \\
&\geq \int\int_{|z-z_0|<\frac{1}{2}(1-|z_0|)} [1 - (\frac{1 + |z_0|}{2})^2)]^{q-2} |Q(z)|^p d\sigma \\
&\geq (\frac{3 + |z_0|}{2})^{q-2} \cdot (\frac{1 - |z_0|}{2})^q \cdot \pi |Q(z_0)|^p \\
&\geq \pi (\frac{1 - |z_0|}{2})^q |Q(z_0)|^p.
\end{aligned}$$

Noticing that for $|z_0| \leq \frac{1}{2}$ we have

$$(\frac{1 - |z_0|}{2})^q \geq (\frac{1}{4})^q = (\frac{1}{4})^s, \quad s = \max\{q, 2\}$$

so we have

$$|Q(z_0)|^p \leq \frac{4^s}{\pi} \|Q(a)\|^p,$$

when $p > 0, q \geq 2$.

If $1 < q < 2$, using (2) to $|z_0| \leq \frac{1}{2}$ we have directly

$$\begin{aligned}
|Q(z_0)|^p &\leq \frac{1}{\pi} (\frac{2}{1 - |z_0|})^2 \int\int_{|z-z_0|<\frac{1}{2}(1-|z_0|)} |Q(z)|^p d\sigma \\
&\leq \frac{4^s}{\pi} \int\int_{|z-z_0|<\frac{1}{2}(1-|z_0|)} (1 - (|z|^2)^{q-2}) |Q(z)|^p d\sigma \\
&\leq \frac{4^s}{\pi} \|Q(z)\|^p.
\end{aligned}$$

The proof is complete.

Lemma 2. *If a function $g(z)$ is analytic and bounded by M in a closed disc $|z| \leq T$, where $T > 1$, we have*

$$p(g, Z) \leq (\frac{\pi}{q-1})^{\frac{1}{p}} \cdot \frac{MT}{T-1} \cdot \exp(-\frac{T-1}{T+1}\lambda)$$

for $p \geq 1$, and $q > 1$.

Proof. We know that the rational function with poles $\zeta \in Z$ and interpolating $g(z)$ at $\frac{1}{\zeta}$ can be expressed by (see [8])

$$r(z) = \frac{1}{2\pi i}\int_{|w|=T>1}[1 - \frac{B(w)}{B(z)}]\frac{g(w)}{w-z}dw, \quad |z| \leq 1.$$

It is easy to see that $r(z) \in R(Z)$ and that

$$g(z) - r(z) = \frac{1}{2\pi i}\int_{|w|=T>1}\frac{B(w)}{B(z)} \cdot \frac{g(w)}{w-z}dw, \quad |z| \leq 1.$$

Hence, the extended Minkowski's inequality gives

$$\begin{aligned}
\rho(g, Z) &\leq \|g(z) - r(z)\| \\
&\leq \{\int\int_{|z|<1}(1-|z|^2)^{q-2}[\frac{1}{2\pi}\int_0^{2\pi}|\frac{B(Te^{i\varphi})}{B(z)} \cdot \frac{g(Te^{i\varphi})}{Te^{i\varphi}-z}|Td\varphi]^p d\sigma_z\}^{\frac{1}{p}} \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}[\int\int_{|z|<1}(1-|z|^2)^{q-2}|\frac{B(Te^{i\varphi})}{B(z)} \cdot \frac{Tg(Te^{i\varphi})}{Te^{i\varphi}-z}|^p d\sigma_z]^{\frac{1}{p}}d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi}T|B(Te^{i\varphi})| \cdot |g(Te^{i\varphi})| \qquad (3) \\
&\quad \cdot\{\int\int_{|z|<1}(1-|z|^2)^{q-2}[|B(z)| \cdot |Te^{i\varphi}-z|]^{-p}d\sigma_z\}^{\frac{1}{p}}d\varphi \\
&\leq \frac{M}{2\pi} \cdot \frac{T}{T-1} \cdot (\frac{\pi}{q-1})^{\frac{1}{p}} \cdot \int_0^{2\pi}|B(Te^{i\varphi})|d\varphi
\end{aligned}$$

From [7] we know that

$$|B(w)| \leq \exp(-\frac{T-1}{T+1}\lambda), \quad |w| = T > 1, \qquad (4)$$

Combining (3) with (4) gives

$$\begin{aligned}
\rho(g, Z) &\leq \frac{M}{2\pi} \cdot \frac{T}{T-1} \cdot (\frac{\pi}{q-1})^{\frac{1}{p}} \cdot \int_0^{2\pi}\exp(-\frac{T-1}{T+1}\lambda)d\varphi \\
&= (\frac{M}{q-1})^{\frac{1}{p}} \cdot \frac{MT}{T-1} \cdot \exp(-\frac{T-1}{T+1}\lambda).
\end{aligned}$$

The proof of Lemma 2 is completed.

To prove the Theorem of this paper, we will use the following simple fact: If $P_N(z)$ is a polynomial of degree at most N and $P_N(z)$ satisfies the inequality

$$|P_N(z)| \le L, \quad \text{if } |z| = r,$$

then it satisfies the inequality

$$|P_N(z)| \le R^N L, \quad \text{if } |z| = Rr > r.$$

$$B(z) = \prod_{\zeta\in Z} \frac{z-\zeta}{1-\zeta z}.$$

Proof of the Theorem of this paper: In [2] we have already proved that for any function $f(z) \in H_q^p(p \ge 1, q > 1)$ and any natural number N there exists a polynomial $P_N(z)$ of degree at most N statisfying the inequality

$$\|f(z) - P_N(z)\| \le A_2 \int_0^{\frac{1}{N}} \frac{\omega(t,f)}{t} dt,$$

where A_2 is a constant independent of N and f. Thus we have

$$\begin{aligned} \rho(f,Z) &\le \|f(z) - P_N(z)\| + \rho(P_N, Z) \\ &\le A_2 \int_0^{\frac{1}{N}} \frac{\omega(t,f)}{t} dt + \rho(P_N, Z). \end{aligned}$$

It is evident that we can suppose $P_N(z)$ satisfies the condition

$$\|P_N(z)\| - \|f(z)\| \le \|f(z) - P_N(z)\| \le \|f(z) - 0\| = \|f(z)\|.$$

Hence we have

$$\|P_N(z)\| \le 2\|f(z)\|.$$

From Lemma 1 for $|z| \le \frac{1}{2}$ we can obtain

$$|P_N(z)| \le (\frac{4^s}{\pi})^{\frac{1}{p}} \cdot \|P_N(z)\| \le (\frac{4^s}{\pi})^{\frac{1}{p}} \cdot 2\|f(z)\|.$$

in which $s = \max\{2, q\}$. Furthermore, we can obtain

$$|P_N(z)| \le (2T)^N \cdot (\frac{4^s}{\pi})^{\frac{1}{p}} \cdot 2\|f(z)\|$$

for $|z| = 2T \cdot \frac{1}{2} = T > 1$. Using Lemma 2, we have

$$\begin{aligned}
\rho(P_N, Z) &\leq (\frac{\pi}{q-1})^{\frac{1}{p}} \cdot (2T)^N \cdot (\frac{4^s}{\pi})^{\frac{1}{p}} \cdot 2\|f(z)\| \\
&\quad \cdot \frac{T}{T-1} \exp(-\frac{T-1}{T+1}\lambda) \\
&= (\frac{4^s}{q-1})^{\frac{1}{p}} \cdot 2\|f(z)\| \cdot \frac{T}{T-1} \exp(-\frac{T-1}{T+1}\lambda \\
&\quad + N \log 2T).
\end{aligned}$$

Particularly, taking $T = 3, N = [\frac{\lambda}{5}] + 1$, we find that

$$\begin{aligned}
\rho(P_N, Z) &\leq (\frac{4^s}{q-1})^{\frac{1}{p}} \cdot \|f(z)\| \cdot \frac{3}{2} \exp(-\frac{1}{2}\lambda + \frac{\lambda}{5} \log 6 + \log 6) \\
&\leq (\frac{4^s}{q-1})^{\frac{1}{p}} \cdot 18\|f(z)\| \cdot \exp(-\frac{\lambda}{10}).
\end{aligned}$$

Noting $\frac{1}{N} \leq \frac{10}{\lambda}$ in this moment, we have

$$\begin{aligned}
A_2 \int_0^{\frac{1}{N}} \frac{\omega(t,f)}{t} dt &\leq A_2 \int_0^{\frac{10}{\lambda}} \frac{\omega(t,f)}{t} dt \\
&= A_2 \int_0^{\frac{1}{\lambda}} \frac{\omega(10t,f)}{t} dt \\
&\leq 10A_2 \int_0^{\frac{1}{\lambda}} \frac{\omega(t,f)}{t} dt.
\end{aligned}$$

Finally, taking $A_1 = 10A_2$, $C = (\frac{4^s}{q-1})^{\frac{1}{p}} \cdot 18\|f(z)\|$ in which $s = \max\{2, q\}$, we can obtain the desired inequality

$$\rho(f, z) \leq A_1 \int_0^{\frac{1}{\lambda}} \frac{\omega(t,f)}{t} dt + Ce^{-\frac{\lambda}{10}},$$

where A_1 is a constant independent of λ and f, C is a constant independent of λ.

The proof is complete.

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