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A COMPARISON OF TOPOLOGIES ON FREE GROUPS

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1. Introduction

Let $F(X)$ denote algebraically the free group generated by the points of the set $X - \{p\}$ where (X,p) is a pointed topological spaces, and let $inc : (X,p) \to (F(X),e)$ be a map of pairs that includes X into $F(X)$ with e the identity element of $F(X)$. In 1948, Graev [5] developed the concept of a free topological group by endowing $F(X)$ with the finest group topology making the map inc a continuous function. This topology is called the Graev topology.

Obviously the Graev topology exists. Describing this topology in concerte terms is in general a difficult matter. Ordman [8] described a topology on $F(X)$ which is the Graev topology whenever X is a k_ω-space. However this topology will not be the Graev topology in general. As an example, it is pointed out in [4] that Ordman's topology is different from the Graev topology when X is homeomorphic to the rationals.

Joiner [7] described a topology which agrees with the Graev topology on various subsets of $F(X)$. Yet this topology is known to, in general, not be a description of the Graev topology. In [9] a very complete description of final topologies on groups is given. While this answer is complete, it suffers by giving a neighborhood filter description when applied to the Graev question as opposed to a basis of open sets. More recently, the authors [2] described a method for constructing the Graev topology.

The purpose of this paper is to compare the tractible topologies described by Ordman, Joiner, and the authors on $F(X)$ and to see which comes closet to describing the Graev topology. In order to accomplish this, we will need to discuss the concept of a semicontinuous group, or

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what Fuchs [6] calls a semitopological group. We shall also obtain a new invariant for pointed topological spaces.

We define the following notations. If G is a group and t some topology on G, let (G^n, t^n) denote the Cartesian product of n copies of G, with the product topology generated by placing the topology t on each copy of G. Let $m_n : G^n \to G$ denote the map defined by $m_n(x_1, x_2, \cdots, x_n) = x_1 x_2 \cdots x_n$. We also declare $m = m_2$.

There is a natural embedding of (G^n, t^n) into (G^{n+1}, t^{n+1}) by mapping $(x_1, x_2, \cdots, x_n) \in G^n$ to $(x_1, x_2, \cdots, x_n, e) \in G^{n+1}$. We will denote the group $\bigcup_{n=1}^{\infty} G^n = G^\infty$. If the product topology t^n is placed on $G^n \subseteq G^\infty$ and the topology coherent with these subspaces is placed on G^∞, we shall denote the resulting topology as t^∞. Finally we define $m_\infty(x) = m_n(x)$ if $x \in G^n \subseteq G^\infty$.

The quotient topology generated on G by the map $m : (G^2, t^2) \to G$ is denoted as $q(t)$. This "process" can be repeated. Let $q_{\alpha+1}(t) = q(q_\alpha(t))$ for successor ordinals and $q_\beta(t) = \bigcap_{\alpha<\beta} q_\alpha(t)$ for limit ordinals.

2. Semicontinuous Groups

A topology S on G is said to be semicontinuous if and only if inversion from (G, S) to (G, S) is continuous and $m : (G^2, S^2) \to (G, S)$ is continuous in each variable separately. Clay [1] has shown that if (G, S) is semicontinuous then $(G, q(S))$ is also semicontinuous. She has also shown that an arbitrary product of semicontinuous groups is a semicontinuous group and that the multiplication map $m : (G^2, S^2) \to (G, q(S))$ is an open map.

If t is a topology on G we will let $S(t)$ denote the finest semicontinuous topology on G contained in t and $g(t)$ denote the finest group topology on G contained in t. For any topology t on G we can define $t^{-1} = \{U | U^{-1} \in t\}$. Clearly $S(t) = S(t \cap t^{-1})$ and $g(t) = g(t \cap t^{-1})$. Since inversion is continuous on $(G, t \cap t^{-1})$ we will without loss of generality assume for the rest of this paper that $t = t \cap t^{-1}$.

The description of $g(t)$ can be viewed as a generalization of the Graev question. The description of $S(t)$ can be viewed as a semicontinuous analog. As we shall see, finding a concrete description of $S(t)$ is not difficult.

Proposition 1. *If T is the quotient topology generated on G by $m_3 : (G^3, D \times t \times D) \to (G, T)$ where D is the discrete topology, then T is semicontinuous.*

Proof. Define $f : G^3 \to G^3$ by $f(a,b,c) = (c^{-1}, b^{-1}, a^{-1})$. Clearly, f is a homeomorphism since inversion is continuous on (G,t). The following commutative diagram shows that inversion is continuous on (G,T).

$$\begin{array}{ccc} (G^3, D \times t \times D) & \xrightarrow{f} & (G^3, D \times t \times D) \\ \Big\downarrow{\scriptstyle m_3} & & \Big\downarrow{\scriptstyle m_3} \\ (G,T) & \xrightarrow{inv} & (G,T) \end{array}$$

Now define $f : (G^3, D \times t \times D) \to (G^3, D \times t \times D)$ by $f(a,b,c) = (xa, b, c)$. Consider the following commutative diagram where $m_x : G \to G$ is the translation $m_x(y) = xy$.

$$\begin{array}{ccc} (G^3, D \times t \times D) & \xrightarrow{f} & (G^3, D \times t \times D) \\ \Big\downarrow{\scriptstyle m_3} & & \Big\downarrow{\scriptstyle m_3} \\ (G,T) & \xrightarrow{m_x} & (G,T) \end{array}$$

Clearly m_x is continuous. A similar argument shows that multiplication on the right by x is also continuous.

We are now in a position to obtain two different concrete descriptions of $S(t)$. We define $t((a,b))$ to be the translated topology on G defined by

$$t((a,b)) = \{V \subseteq G | a^{-1} V b^{-1} \in t\}.$$

Proposition 2. $S(t) = T = \bigcap_{a,b \in G} t((a,b))$.

Proof. $U \in T$ if and only if $m_3^{-1}(U)$ is open in $(G^3, D \times t \times D)$. But $m_3^{-1}(U)$ is open if and only if $m_3^{-1}(U) \cap \{(a,g,b) | g \in G\}$ is open for every fixed pair $a, b \in G$. Since $m_3^{-1}(U) \cap \{(a,g,b) | g \in G\} = \{(a,g,b) | (g \in a^{-1} U b^{-1}\}$ we have that $U \in T$ if and only if $U \in \bigcap_{a,b \in G} t((a,b))$.

By Proposition 1 we have that $T \subseteq S(t)$. Since $S(t)$ is semicontinuous, we have that for every $a, b \in G$ and for every $U \in S(t)$, $a^{-1} U b^{-1} \in S(t)$. Hence since $S(t) \subset t$ we have $S(t) \subseteq \bigcap_{a,b \in G} t((a,b))$.

Proposition 3. $q_2(t) \subseteq S(t)$.

Proof. We remember that m and hence $m \times m$ are open maps. The following diagram proves the proposition:

$$\begin{array}{ccc}
(G^4, D \times t \times D \times D) & \xrightarrow{id} & (G^4, t^4) \\
\downarrow{\scriptstyle m\times m} & & \downarrow{\scriptstyle m\times m} \\
(G^2, \cap_{a\in G} t((a,e)) \times D) & \xrightarrow{id} & (G^2, q(t) \times q(t)) \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\
(G, S(t)) & \xrightarrow{id} & (G, q_2(t))
\end{array}$$

Corollary 4. $q_\omega(S(t)) = q_\omega(t)$.

Proof. For all $n \in \mathbf{N}$ we have $q_{n+2}(t) \subseteq q_n(S(t)) \subseteq q_n(t)$.

3. A Few Comparisons

Ordman's topology [8], θ is defined using a k-product on X^n instead of a product topology. We denote this k-product topology by t^n_k. We define X^∞, t^∞_k, and t^∞ in the same manner as section 1.

Since we will be discussing topologies on generating sets X of the free group $F(X)$ instead of a topology on the entire group, we will need to extend the topology from X to $F(X)$. A natural extension is the topology $\{U \subseteq F(X) | U \cap X \in t\}$. We shall abuse the notation and call this topology on $F(X)$ by the name t also. By using the same ideas described in section 2, we may also assume that inversion from $(F(X), t)$ to $(F(X), t)$ is continuous.

Proposition 5. $q_\omega(t) \subseteq \theta$.

Proof. First we wish to show that the map $m_\infty : (G^\infty, (S(t))^\infty) \to (G, q_\omega(t))$ is an open map. A straightforward induction argument shows that $m_{(2^n)} : (G^{(2^n)}, (S(t))^{(2^n)}) \to (G, q_n(S(t)))$ is an open map.

Let $B \in (S(t))^\infty$. Then we have $m_\infty(B) = \bigcup_{n=1}^\infty m_{(2^n)}(B \cap G^{(2^n)})$. If $U(n) = m_{(2^n)}(B \cap G^{(2^n)})$, then we have that $U(n) \in q_n(S(t))$. But this means that $m_\infty(B) \in q_\omega(S(t)) = q_\omega(t)$ since $m_\infty(B) = \bigcup_{j=m}^\infty U(j)$ for all $m \in \mathbf{N}$, $U(n) \subseteq U(n+1)$ for all $n \in \mathbf{N}$, and $q_{n+1}(S(t)) \subseteq q_n(S(t))$ for all

$n \in \mathbf{N}$.

Consider the following commutative diagram:

$$\begin{array}{ccccccc}
(X^\infty, t_k^\infty) & \xrightarrow{id} & (X^\infty, t^\infty) & \xrightarrow{inc} & ((F(X))^\infty, t^\infty) & \xrightarrow{id} & ((F(X))^\infty, (S(t))^\infty) \\
\big\downarrow m_\infty & & & & & & \big\downarrow m_\infty \\
(F(X), \theta) & & & \xrightarrow{\qquad\qquad id \qquad\qquad} & & & (F(X), q_\omega(t))
\end{array}$$

Corollary 6. *If (X,t) is a k_ω-space then $q_w(t)$ is a group topology.*

Proof. This follows from the work of Ordman [8] and Proposition 5. By [2] we know that $g(t) \subseteq q_n(t)$ for all $n \in \mathbf{N}$.

Corollary 7. *If J is Joiner's topology for $F(X)$ [7], then $q_\omega(t) \subseteq J$.*

4. The Free Index

Let $h : G \to G'$ be an onto homomorphism and suppose that (X,t) is a subset of G, (X',t') is a subset of G' and that $h|_X$ is a homeomorphism from (X,t) to (X',t'). As before we extend the topology t from X to G by declaring $U \subseteq G$ to be open if and only if $U \cap X \in t$. We likewise extend the topology t' from X' to G' and abuse notation by calling these extended topologies t and t'. Also as before we assume that inversion is continuous on (G,t) and on (G',t').

Proposition 8. *If $(G, q_\alpha(t))$ is a topological group, then $(G', q_\alpha(t'))$ is a topological group.*

Proof. Let T be the quotient topology on G' for $h : (G, q_\alpha(t)) \to (G', T)$. Since $q_\alpha(\tau)$ is a group topology, T is a group topology. Since $h|_X$ is a homeomorphism, we have that inclusion from (X',t') into (G',t') is continuous. Hence $T \subseteq g(t')$.

We have that $h : (G,t) \to (G',t')$ is continuous. The following diagram shows that h is continuous for every successor ordinal:

$$\begin{array}{ccc} (G^2,(q_\alpha(t))^2) & \xrightarrow{h\times h} & ((G')^2,(q_\alpha(t'))^2) \\ \big\downarrow m & & \big\downarrow m' \\ (G,q_{\alpha+1}(t)) & \xrightarrow{h} & (G',q_{\alpha+1}(t')) \end{array}$$

Since $q_\beta(t) = \bigcap_{\alpha<\beta} q_\alpha(t)$ whenever β is a limit ordinal, a transfinite induction shows that h is continuous for every ordinal. Since T is the quotient topology on G' generated by h, we have that $q_\alpha(t') \subseteq T$. But for every ordinal γ we have $g(t') \subseteq q_{\gamma+1}(t') \subseteq q_\gamma(t')$. Thus $T = q_\alpha(t')$.

It should be noted that $h|_X$ need not be a homeomorphism in order for Proposition 8 to hold. Any map with a continuous right inverse would suffice.

Let (X,t) be a space and $inc : (X,p) \to (F(X),e)$ the map of pairs described in the introduction. By [2] we know that there is a first ordinal α such that $q_\alpha(t) = g(t)$. We call this ordinal the *free index* of (X,t). By the nature of $F(X)$ and Proposition 8, we have that the free index is an upper bound for the number of iterations of the "q-operator" on any group G that is generated by X, in order to transform G into a topological group. By Corollary 6, the free index of a k_ω-space is bounded by ω.

Proposition 9. *Let t_1 and t_2 be topologies on the groups G_1 and G_2. Then $g(t_1) \times g(t_2) = g(t_1 \times t_2)$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} & ((G_1\times G_2)^2,(t_1\times t_1)^2) & \\ {}^{m}\swarrow & & \searrow {}^{m} \\ (G_1\times G_2, q(t_1\times t_2)) & \xrightarrow{id} & (G_1\times G_2, q(t_1)\times q(t_2)) \end{array}$$

This shows that $q(t_1) \times q(t_2) \subseteq q(t_1 \times t_2)$. Therefore $g(t_1) \times g(t_2) \subseteq g(t_1 \times t_2)$. Let $t_1' \times \{e\}$ be the relative group topology on $G_1 \times \{e\} \subseteq (G_1 \times G_2, g(t_1 \times t_2))$ and $\{e\} \times t_2'$ be relative group topology on $\{e\} \times G_2 \subseteq (G_1 \times G_2, g(t_1 \times t_2))$. By [3] we know that the finest group topology on $G_1 \times G_2$ which simultaneously extends $t_1' \subseteq g(t_1)$ since the identity map

$(G_1 \times \{e\}, q(t_1 \times t_2)) \xrightarrow{id} (G_1 \times \{e\}, q(t_1))$ is continuous. Likewise $t_2' \subseteq g(t_2)$. Therefore $g(t_1 \times t_2) \subseteq t_1' \times t_2' \subseteq g(t_1) \times g(t_2)$.

Proposition 10. (a) *If t is coarser than the particular point topology, then the free index of t is one.*

(b) *If X is an infinite set and t is the finite complement topology, then the free index of t is two.*

Proof. (a) We have that $q(t) \subseteq [\bigcap_{x \in G} t((x,e))] \cap [\bigcap_{x \in G} t((e,x))]$. If every nonempty open set in t contains a specific point, then $q(t)$ is clearly the indiscrete topology.

(b) Let $x \in X_0 = X \cup_e X^{-1} \subseteq F(X)$, where X_0 is as defined in [8]. Let $x \in U \in q(t)$. Then $(x,e) \in m^{-1}(U)$ is open in $F(X) \times F(X)$. We can find a basic open set $W_1 \times W_2$ in the relative topolgy on $X_0 \times X_0 \subseteq F(X) \times F(X)$ that contains (x,e) and is contained in $m^{-1}(U)$. Since both W_1 and W_2 are all but finitely many points of X_0 and X_0 is infinite, we have that $m^{-1}(e) \cap (W_1 \times W_2)$ is nontrivial. Thus the relative topology on X_0 is coarser than the particular point topology in $q(t)$ and e fails to be a closed set. However x^3 is closed. Hence $q(t)$ fails to be a group topology. Hence by (a), $q_2(t)$ is the indiscrete topology.

Proposition 11. *If X is a Hausdorff space that contains a convergent nonconstant sequence, then the free index of X is greater than one.*

Proof. Let Y be the complement of the convergent sequence in X. If Y is finite, let $G = \{e\}$. If Y is infinite, we let Γ be a set of ordinals with the same cardinality as X. Let $G = \oplus_{\alpha \in \Gamma} \mathbf{Z}_\alpha$. Consider the group $G \times \mathbf{Z}$. We can find a bijection $f : X \to G \times \mathbf{Z}$ which maps the nonconstant sequence of X onto $\{e\} \times \mathbf{Z}$. If we define $t = \{f(U) | U \text{ is open in } X\}$, then we will have a nongroup topology on $\{e\} \times \mathbf{Z}$ contained in $G \times \mathbf{Z}$. Hence $g(t)$ has a nontrivial normal subgroup $\{\overline{(e,0)}\}$. However $(e,0)$ remains closed in $(G \times \mathbf{Z}, q(t))$. Therefore, by Proposition 8, we are done.

Proposition 12. *Let X have the topology of a nonconstant convergent sequence. Then $(F(X), q_n(t))$ fails to be a topological group for any $n \in \mathbf{N}$.*

Proof. Let $A(X)$ denote the free abelian group generated by X. We may assume that e is the point of convergence for X (i.e. $\{x_n\}_{n=1}^{\infty} \to e$). Let $V = \{w | w = x_{n_1}^{\epsilon_1} X_{n_2}^{\epsilon_2} \cdots x_{n_k}^{\epsilon_k}$ with $\epsilon_i = \pm 1$ for $1 \leq i \leq k$ when w is in reduced form, or $w = e\}$. As usual we include X into $A(X)$ and extend the topology from X to $A(X)$. Also as usual we assume that the extended topology t makes inversion on $A(X)$ a continuous function.

By Proposition 2 we have that $S(t) = \bigcup_{a \in A(X)} t((a, e))$. We note that $V \in S(t)$. Since $m_{(2^n)} : (A^{(2^n)}(X), (S(t))^{(2^n)}) \to (A(X), q_n(S(t)))$ is an open map, we have that $V^{(2^n)} \in q_n(S(t))$. But $q_n(S(t))$ cannot be a topological group since $x_m^{(2^n)+1} \notin V^{(2^n)}$ for all $m \in \mathbf{N}$ yet $\{x_m^{(2^n)+1}\}_{n=1}^{\infty} \to e$ in $g(t)$.

Corollary 13. *The free index for a nonconstant convergent sequence is* ω.

We conclude this paper by noting that the pairs (G, X) and maps of pairs $h : (G, X) \to (G', X')$ where h is a group homomorphism and $h|_X$ is continuous, form a category. The proof of Proposition 8 demonstrates that $h : G \to G'$ is continuous at each stage of the "q-operator". Hence the process actually describes a functor to the category of topological groups and continuous group homomorphisms, and defines an index for each pair.

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