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PRIME IDEALS ASSOCIATED TO THE I-ADIC FILTRATION *

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1. Introduction

In this paper, R will always be a commutative Noetherian ring, I will be an ideal of R and M will be a finitely generated R-module unless otherwise stated. We denote by $\mathbf{R}$ the Rees ring $R[u, It]$ of R with respect to I, where t is an indeterminate and $u = t^{-1}$. Similarly, we will denote by $\mathbf{M}$ the graded $\mathbf{R}$-module $M[u, It] = \bigoplus_{n\geq 0} I^n M$. It is finitely generated over $\mathbf{R}$. If (R, P) is a local ring, then R^* will be the P-adic completion of R and M^* will be the completion of M with respect to the P-adic filtration $\{P^n M\}_{n\geq 0}$. In particular, we will denote by R_P^* and M_P^* the P_P-adic completion of R_P and M_P, respectively, where R_P, P_P and M_P are localizations of R, P and M at P, respectively.

In recent years, two concepts similar to regular sequences have appeared, asymptotic sequences and essential sequences. In several papers, [[3], [4], [5], [6], [7], [9], [14], [15]], D.Katz, S. McAdam and L.J.Ratliff, Jr. have introduced various sets of prime ideals associated to ideals of Noetherian rings. We denote these by $\mathbf{E}(I, R)$, $\mathbf{Q}(I, R)$, $\bar{\mathbf{A}}^*(I, R)$, and $\bar{\mathbf{Q}}^*(I, R)$. For the definition of these, see definition (2.1). It was shown that these sequences behave very similarly to regular sequences. For a detailed survey of the prime ideals associated to the I-adic filtration, we refer the reader to the above papers.

The following result about their behavior under lifting and contraction played very important role in the asymptotic theory.

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Theorem 1.1. *Let R and T be Noetherian rings with $R \subseteq T$ a faithfully flat extension of R. Let $A(I,R)$ represent any one of* $\mathbf{E}(I,R)$, $\mathbf{Q}(I,R)$, $\bar{\mathbf{A}}^*(I,R)$, *or* $\bar{\mathbf{Q}}^*(I,R)$.

(1) *If $Q \in A(IT,T)$, then $Q \bigcap R \in A(I,R)$.*

(2) *If $P \in A(I,R)$ and if $Q \in Spec(T)$ is minimal over PT, then*

$$Q \in A(IT,T).$$

The purpose of the present paper, we extend this result on the contraction and lifting of asymptotic primes to modules.

2. Preliminaries

We begin this section by defining the counterparts for modules of the sets $\bar{\mathbf{Q}}^*(I,M)$, $\bar{\mathbf{A}}^*(I,M), \mathbf{Q}(I,M)$ and $\mathbf{E}(I,M)$ and prove a few of the basic properties of these sets.

Definition 2.1. Let R be a commutative noetherian ring with unity and I an ideal. Let M be a finitely generated R-module.

(1) $\bar{\mathbf{Q}}^*(I,M) = \{P \in Spec(R)$; there exist a prime q minimal in $Supp_{R_P^*}(M_P^*)$ with PR_P^* minimal over $IR_P^* + q\}$

(2) $\bar{\mathbf{A}}^*(I,M) = \{P :\ P = Q \cap R$ for some $Q \in \bar{\mathbf{Q}}^*(u\mathbf{R},\mathbf{M})\}$

(3) $\mathbf{Q}(I,M) = \{P \in Spec(R)$:there exists a prime $q \in Ass_{R_P^*}(M_P^*)$ with P_P^* minimal over $IR_P^* + q\}$

(4) $\mathbf{E}(I,M) = \{P : P = Q \cap R$ for some $Q \in \mathbf{Q}(u\mathbf{R},\mathbf{M})\}$

Lemma 2.2. *Let $A(I,M)$ represent any one of the above. Let S be a multiplicatively closed subset of R disjoint from the prime P. Then*

$$P \in A(I,M) \quad \textit{if and only if} \quad P_S \in A(I_S,M_S)$$

Proof. This is straight forward.

We state the following important theorem which is sometimes known as Bourbaki's Theorem. [[1], Chap.6, Section 2.6, Theorem 2]. It and its corollary will be used frequently in this paper.

Theorem 2.3 (Bourbaki). *Let $\Phi : R \to T$ be a ring homomorphism of Noetherian rings and M a R-module. Suppose that F is a T-module which is flat as an R-module.*

(1) *For any prime ideal P of R*

$$\Phi^{-1}(Ass_T(F/PF)) = Ass_R(F/PF) = \begin{cases} \{P\} & if F/PF \neq 0 \\ \emptyset & if F/PF = 0 \end{cases}$$

(2) $Ass_T(M \otimes_R F) = \bigcup_{P \in Ass_R(M)} Ass_T(F/PF)$

The next corollary is obvious from theorem (2.3). But we state this for the convenience of readers.

Corollary 2.4. *Let (R, m) be a Noetherian local ring and R^* be its m-adic completion. Let M be a R-module and M^* its completion with respect to the m-adic filtration $\{\mathrm{m}^n M\}_{n \geq 0}$. Then*

$$Ass_{R^*}(M^*) = \bigcup_{P \in Ass_R(M)} Ass_{R^*}(R^*/PR^*)$$

Proof. Set $R^* = T = F$. The corollary follows from Theorem (2.3).

The next lemma immediately follows from the definition and lemma (2.2) but we give a proof of this.

Lemma 2.5. *If $q \in Ass_R(M)$ and P is a prime ideal minimal over $I + q$ then $P \in \mathbf{Q}(I, M)$. If q is a minimal in $Supp_R(M)$ then $P \in \bar{\mathbf{Q}}^*(I, M)$.*

Proof. Let $q \in Ass_R(M)$ and let P be a minimal prime over $I + q$. By lemma (2.2), we may assume that (R, P) is local. Let $q^* \in Spec(R^*)$ be minimal over qR^*. Then $q^* \in Ass_{R^*}(R^*/qR^*)$ and P^* is minimal over $IR^* + q^*$. By corollary (2.4), $q^* \in Ass_{R^*}(M^*)$. Hence $P \in \mathbf{Q}(I, M)$. If q is minimal in $Supp_R(M)$, then q^* is minimal in $Supp_{R^*}(M^*)$. Hence $P \in \bar{\mathbf{Q}}^*(I, M)$.

3. Main Results

In this section, we extend the result on the contraction and lifting of asymptotic primes to modules.

First, we prove the following theorem which is one of our main theorems. It shows that most information concerning the primes associated to an ideal are useful for modules.

Theorem 3.1. *Let R be a commutative noetherian ring with unity and I an ideal. Let M be a finitely generated R-module.*

(1) $P \in \mathbf{Q}(I,M)$ *(resp.* $\mathbf{E}(I,M)$*) if and only if there exists* $q \in Ass_R(M)$ *with* $q \subseteq P$ *and*

$$P/q \in \mathbf{Q}(I+q/q, R/q) \text{ (resp. } \mathbf{E}(I+q/q, R/q)).$$

(2) $P \in \bar{\mathbf{Q}}^*(I,M)$ *(resp.* $\bar{\mathbf{A}}^*(I,M)$*) if and only if there exists a prime q minimal in* $Supp_R(M)$ *with* $q \subseteq P$ *and*

$$P/q \in \bar{\mathbf{Q}}^*(I+q/q, R/q) \text{ (resp. } \bar{\mathbf{A}}^*(I+q/q, R/q)).$$

Proof. By lemma (2.2), we may assume that (R,P) is local.

The proof of (1): Let $P \in \mathbf{Q}(I,M)$. Then there exists

$$q^* \in Ass_{R^*}(M^*) \text{ with } P^* \text{ minimal over } IR^* + q^*.$$

By corollary (2.4),

$$q^* \in Ass_{R^*}(R^*/qR^*) \text{ where } q = q^* \cap R \in Ass_R(M).$$

Hence $q^*/qR^* \in Ass_{R^*/qR^*}(R^*/qR^*)$ and P^*/qR^* is minimal over $((IR^*+qR^*)/qR^*) + q^*/qR^*$. Therefore, $P/q \in \mathbf{Q}(I+q/q, R/q)$.

Conversely, let $q \in Ass_R(M)$ with $P/q \in \mathbf{Q}(I+q/q, R/q)$. Then there exists $q^* \in Spec(R^*)$ such that $q^*/qR^* \in Ass_{R^*/qR^*}(R^*/qR^*)$ and such that P^*/qR^* is minimal over $((IR^*+qR^*)/qR^*) + q^*/qR^*$. Then $q^* \in Ass_{R^*}(R^*/qR^*) \subseteq Ass_{R^*}(M^*)$ and P^* is minimal over $IR^* + q^*$. Thus $P \in \mathbf{Q}(I,M)$. Let $P \in \mathbf{E}(I,M)$. By the definition, $Q \cap R = P$ for some $Q \in \mathbf{Q}(u\mathbf{R},\mathbf{M})$. By the above, there exists $q \in Ass_{\mathbf{R}}(\mathbf{M})$ such that $q \subseteq Q$ and $Q/q \in \mathbf{Q}(u\mathbf{R}+q/q, \mathbf{R}/q)$. Set $p = q \cap R$, then $p \in Ass_R(M)$ and $q = pR[u,t] \cap \mathbf{R}$. But we have an isomorphism $\mathbf{R}/q \cong (R/p)[((I+p)/p)t, u]$ [see [12]]. Under this isomorphism, Q/q intersects R/p at P/p. Hence $P/p \in \mathbf{E}(I+p/p, R/p)$ Conversely, let $q \in Ass_R(M)$ with $P/q \in \mathbf{E}(I+q/q, R/q)$ and let $p = qR[u,t] \cap \mathbf{R}$. Then there is a $Q/p \in \mathbf{Q}(u\mathbf{R}+p/p, \mathbf{R}/p)$ such that Q/p intersects R/q at P/q under the isomorphism $\mathbf{R}/p \cong (R/q)[(I+q/q)t, u]$. Since $p \in Ass_{\mathbf{R}}(\mathbf{M})$, by the above, $Q \in \mathbf{Q}(u\mathbf{R},\mathbf{M})$. Thus $P \in \mathbf{E}(I,M)$.

The proof of (2): Let $P \in \bar{\mathbf{Q}}^*(I,M)$. Then there exists a prime q^* minimal in $Supp_{R^*}(M^*)$ with P^* minimal over $IR^* + q^*$. Set $q = q^* \cap R$, then

q is minimal in $Supp_R(M)$,and q^*/qR^* is minimal in $Supp_{R^*/qR^*}(R^*/qR^*)$, and P^*/qR^* is minimal over IR^*+q^*/qR^*. Hence $P/q \in \bar{\mathbf{Q}}^*(I+q/q, R/q)$. Conversely, let q be a minimal prime ideal in $Supp_R(M)$ with $q \subseteq P$ and $P/q \in \bar{\mathbf{Q}}^*(I+q/q, R/q)$. Then there is a prime q^*/qR^* minimal in $Supp_{R^*}(R^*/qR^*)$ with P^*/qR^* minimal over $((IR^*+qR^*)/qR^*)+q^*/qR^*$. It is clear that q^* is minimal in $Supp_{R^*}(M^*)$ and that P^* is minimal over IR^*+q^*. Therefore $P \in \bar{\mathbf{Q}}^*(I, M)$. Let $P \in \bar{\mathbf{A}}^*(I, M)$. By the definition, there is a $Q \in \bar{\mathbf{Q}}^*(u\mathbf{R}, \mathbf{M})$ with $P = Q \bigcap R$. By the above, there is a prime q minimal in $Supp_{\mathbf{R}}(\mathbf{M})$ with $q \subseteq Q$ and $Q/q \in \bar{\mathbf{Q}}^*(u\mathbf{R}+q/q, \mathbf{R}/q)$. Set $p = q \cap R$. Then p is minimal in $Supp_R(M)$ and $q = pR[t,u] \cap \mathbf{R}$ because $q \in Ass_{\mathbf{R}}(\mathbf{M})$. Under the isomorphism $\mathbf{R}/q \cong (R/p)[(I+p/p)t, u]$, $(Q/q) \cap R/p = P/p$. Hence $P/p \in \bar{\mathbf{A}}^*(I+p/p, R/p)$. Conversely, let q be minimal in $Supp_R(M)$ with $q \subseteq P$ and $P/q \in \bar{\mathbf{A}}^*(I+q/q, R/q)$ and let $q' = qR[t,u] \cap \mathbf{R}$. By the definition, there is a prime $p^\sharp \in \bar{\mathbf{Q}}^*(u(R/q)[(I+q/q)t, u], (R/q)[(I+q/q)t, u)])$ with $p^\sharp \cap (R/q) = P/q$. By the isomorphism $(R/q)[(I+q/q)t, u] \overset{\Phi}{\cong} \mathbf{R}/q'$, $\Phi(p^\sharp) = Q/q' \in \bar{\mathbf{Q}}^*(u\mathbf{R}+q'/q', \mathbf{R}/q')$ with $Q \bigcap R = P$. It is clear that q' is minimal in $Supp_{\mathbf{R}}(\mathbf{M})$. Hence, by the above, $Q \in \bar{\mathbf{Q}}^*(u\mathbf{R}, \mathbf{M})$. Thus $P \in \bar{\mathbf{A}}^*(I, M)$.

Remark 3.2. Recall that if $M = R$ then

$$\bar{\mathbf{A}}^*(I, R) = Ass_R(R/(I^n)_a) \quad \text{for all large } n \quad [\text{ See [11]}].$$

$$\bar{\mathbf{B}}^*(I, R) \subseteq \mathbf{Q}(I, R) \cap \bar{\mathbf{A}}^*(I, R)$$

$$\mathbf{Q}(I, R) \cup \bar{\mathbf{A}}^*(I, R) \subseteq \mathbf{E}(I, R) \subseteq \mathbf{A}^*(I, R)$$

where $\mathbf{A}^*(I, R) = Ass_R(R/I^n)$, for all large n [See [2]]. In particular, all sets in the above are finite.

Corollary 3.3. *Let R be a commutative noetherian ring with unity and I an ideal. Let M be a finitely generated R-module. The following hold.*

(1) $\bar{\mathbf{Q}}^*(I, M) \subseteq \mathbf{Q}(I, M) \bigcap \bar{\mathbf{A}}^*(I, M)$

(2) $\mathbf{Q}(I, M) \bigcup \bar{\mathbf{A}}^*(I, M) \subseteq \mathbf{E}(I, M)$

(3) *The sets $\bar{\mathbf{Q}}^*(I, M)$, $\bar{\mathbf{A}}^*(I, M)$, $\mathbf{Q}(I, M)$ and $\mathbf{E}(I, M)$ are finite.*

Proof. This follows from Definition 2.1, Theorem 3.1 and those results known for rings.

Finally, we extend the result on the contraction and lifting of asymptotic primes to modules.

Theorem 3.4. *Let $A(I,M)$ denote any of $\bar{\mathbf{Q}}^*(I,M)$, $\bar{\mathbf{A}}^*(I,M)$, $\mathbf{A}^*(I,M)$, $\mathbf{Q}(I,M)$ or $\mathbf{E}(I,M)$. Let $\Phi : R \to T$ be a faithfully flat ring homomorphism.*

(1) *If $Q \in A(IT, M \otimes_R T)$, then $Q \bigcap R \in A^*(I,M)$.*

(2) *If $P \in A(I,M)$ and Q is a minimal prime ideal over PT, then $Q \in A(IT, M \otimes_R T)$.*

Proof. If $A(I,M) = \mathbf{A}^*(I,M)$, (1) and (2) follow from Theorem 2.3.

$A(I,M) = \mathbf{Q}(I,M)$: Let $P \in \mathbf{Q}(IT, M \otimes_R T)$. By Theorem 3.1 there is a prime $q \in Ass_T(M \otimes_R T)$ such that $P/q \in \mathbf{Q}(IT + q/q, T/q)$. Let $q' = q \bigcap R$. Then by Theorem 2.3 $q' \in Ass_R(M)$ and $q/q'T \in Ass_{q/q'T}(T/q'T)$. But $P/q \cong (P/q'T)/(q/q'T)$ and thus by Theorem 3.1 $P/q'T \in \mathbf{Q}(IT + q'T/q'T, T/q'T)$. But since $T/q'T$ is faithfully flat over R/q', $(P/q'T) \bigcap R/q' \in \mathbf{Q}(I + q'/q', R/q')$. But it is clear that $(P/q'T) \bigcap R/q' = (P \cap R)/q'$. Hence by Theorem 3.1 $P \bigcap R \in \mathbf{Q}(I,M)$. Conversely, let $P \in \mathbf{Q}(I,M)$ and let Q be minimal over PT. Then there is $q \in Ass_R(M)$ such that $P/q \in \mathbf{Q}(I + q/q, R/q)$. Since Q/qT is minimal over PT/qT and T/qT is faithfully flat over R/q, $Q/qT \in \mathbf{Q}(IT + qT/qT, T/qT)$ [see [10], (1.9)]. Hence there is $q^*/qT \in Ass_{T/qT}(T/qT)$ such that $Q/q^* = (Q/qT)/(q^*/qT) \in \mathbf{Q}(IT + q^*/q^*, T/q^*)$. But it is clear that $q^* \in Ass_T(T/qT)$. By Theorem 2.3 $q^* \in Ass_T(M \otimes_R T)$. Thus $Q \in \mathbf{Q}(IT, T)$.

$A(I,M) = \bar{\mathbf{Q}}^*(I,M)$: Let $P \in \bar{\mathbf{Q}}^*(IT, M \otimes_R T)$ By Theorem 3.1 there is a prime q minimal in $Supp(M \otimes_R T)$ such that $P/q \in \bar{\mathbf{Q}}^*(IT + q/q, T/q)$. Let $q' = q \bigcap R$. Then it is clear that q' is minimal in $Supp(M)$ and that $q/q'T$ is minimal in $Ass_{T/q'T}(T/q'T)$ and thus is in $Supp(T/q'T)$. Hence $P/q'T \in \bar{\mathbf{Q}}^*(IT + q'T/q'T, T/q'T)$. Since $T/q'T$ is faithfully flat over R/q', $(P \cap R)/q' \in \bar{\mathbf{Q}}^*(I + q'/q', R/q')$ [see [10], (1.9)]. Thus $P \bigcap R \in \bar{\mathbf{Q}}^*(I,M)$. Conversely, let $P \in \bar{\mathbf{Q}}^*(I,M)$ and let $Q \in Spec(T)$ be a prime minimal over PT. By Theorem 3.1, there is a prime q minimal in $Supp(M)$ such that $P/q \in \bar{\mathbf{Q}}^*(I + q/q, R/q)$. Since Q/qT is minimal over PT/qT and T/qT is faithfully flat over R/q, $Q/qT \in \bar{\mathbf{Q}}^*(IT + qT/qT, T/qT)$. By Theorem 3.1, there is a minimal prime $q^*/qT \in Ass_{T/qT}(T/qT)$ such that $Q/q^* \in \bar{\mathbf{Q}}^*(IT + q^*/q^*, T/q^*)$. By Theorem 2.3, q^* is minimal in $Supp(M \otimes_R T)$. Therefore, $Q \in \bar{\mathbf{Q}}^*(IT, M \otimes_R T)$.

$A(I,M) = \mathbf{E}(I,M)$: Let $\mathbf{T}$ be the Rees ring of T with respect to IT. Let $Q \in \mathbf{E}(IT, M \otimes_R T)$. Then there exists a prime ideal P in $\mathbf{Q}(u\mathbf{T}, M \otimes_R \mathbf{T})$ with $P \cap T = Q$. It is clear that $\mathbf{T}$ is faithfully flat over

$\mathbf{R}$ [see [10], Lemma 1.8]. Hence $P \cap \mathbf{R} \in \mathbf{Q}(u\mathbf{R}, \mathbf{M})$. Thus $Q \cap R = (P \cap \mathbf{R}) \cap R \in \mathbf{E}(I, M)$. Let $P \in \mathbf{E}(I, M)$ and $Q \in Spec(T)$ a minimal prime over PT. Then there exists a prime ideal p in $\mathbf{Q}(u\mathbf{R}, \mathbf{M})$ such that $p \bigcap R = P$. Let q be a minimal prime over $p\mathbf{T}$ in $\mathbf{T}$, then $q \in \mathbf{Q}(u\mathbf{T}, M \otimes_R \mathbf{T})$. Hence we have $q \cap T \in \mathbf{E}(IT, M \otimes_R T)$. We may assume that (R, P) and (T, Q) are local rings, so that PT is Q-primary ideal and $q \cap T \supseteq PT$. Hence we get $Q = q \bigcap T$. This completes the proof.

$A(I, M) = \bar{\mathbf{A}}^*(I, M)$: The proof is analogous to the case $A(I, M) = \mathbf{E}(I, M)$.

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