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THE MAXIMUM IDEMPOTENT-SEPARATING CONGRUENCE ON AN INVERSE Γ-SEMIGROUP*

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1. Introduction

Let A and B be two non-empty sets, M the set of all mappings from A to B, and Γ a set of some mappings from B to A. The usual mapping product of two elements of M cannot be defined. But if we take f, g from M and α from Γ then the usual mapping product $f\alpha g$ can be defined. Also we find that $f\alpha g \in M$ and $(f\alpha g)\beta h = f\alpha(g\beta h)$ for $f, g, h \in M$ and $\alpha, \beta \in \Gamma$.

If M be the set of $m \times n$ matrices and Γ be a set of some $n \times m$ matrices then we can define $A_{m,n}\alpha_{n,m}B_{m,n}$ such that $(A_{m,n}\alpha_{n,m}B_{m,n})\beta_{n,m}C_{m,n} = A_{m,n}\alpha_{n,m}(B_{m,n} \circ \beta_{n,m}C_{m,n})$ where $A_{m,n}, B_{m,n}, C_{m,n} \in M$ and $\alpha_{n,m}, \beta_{n,m} \in \Gamma$. An algebraic system satisfying associative property of the above type is a Γ-semigroup [1].

Definition. Let $M = \{a, b, c, \cdots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \cdots\}$ be two non-empty sets. M is called a Γ-semigroup if (i) $a\alpha b \in M$ for $\alpha \in \Gamma$ and $a, b \in M$ and (ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in M$ and for all $\alpha, \beta \in T$. A semigroup can be considered as a Γ-semigroup in the following sense. Let S be an arbitrary semigroup. Let 1 be a symbol not representing any element of S. Let us extend the given binary relation in S to $S \cup \{1\}$ by defining $11 = 1$ and $1a = a1 = a$ for all a in S. It can be shown that $S \cup \{1\}$ is a semigroup with identity element 1. Let $\Gamma = \{1\}$. Putting $ab = a1b$ it can be shown that the semigroup S is a Γ-semigroup where $\Gamma = \{1\}$. Since every semigroup is a Γ-semigroup in the above sense the main thrust of our work is to extend different fundamental results of semigroup to Γ-semigroup. In [2], [3], [4] and [5] we have extended some

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results of semigroup to Γ-semigroup. In this paper we establish maximum idempotent separating congruence on an inverse Γ-semigroup.

2. Preliminaries

We recall following definitions and results from [2], [3], [4] and [5].

Definition. Let M be a Γ-semigroup. A non-empty subset B of M is said to be a Γ-subsemigroup of M if $B\Gamma B \subset B$.

Definition. Let M be a Γ-semigroup. An element $a \in M$ is said to be regular if $a \in a\Gamma M\Gamma a$, where $a\Gamma M\Gamma a = \{a\alpha b\beta a : b \in M, \alpha, \beta \in \Gamma\}$. A Γ-semigroup M is said to regular if every element of M is regular.

Definition. Let M be a Γ-semigroup. An element $e \in M$ is said to be an idempotent in M if there exists an $a \in \Gamma$ such that $e\alpha e = e$. In this case we shall say e is an α-idempotent.

Definition. Let M be a Γ-semigroup and $a \in M$. Let $b \in M$ and $\alpha, \beta \in \Gamma$. b is said to be an (α, β) inverse of a if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$. In this case we shall write $b \in V_\alpha^\beta(a)$.

We have defined $\mathcal{L}, \mathcal{R}, \mathcal{H}$, the analogue of Green's relation in [3].

Lemma 2.1. *Let M be a regular Γ-semigroup and let $a \in M$. Suppose e is an α-idempotent and f is a β-idempotent of M with $e\mathcal{R}a\mathcal{L}f$. Then there exists a unique $b \in V_\beta^\alpha(a)$ such that $a\beta b = e$ and $b\alpha a = f$.*

Definition. A regular Γ-semigroup M is called an inverse Γ-semigroup if $|V_\alpha^\beta(a)| = 1$, for all $a \in M$ and for all $\alpha, \beta \in \Gamma$, whenever $V_\alpha^\beta(a) \neq \emptyset$. That is every element a of M has a unique (α, β) inverse whenever (α, β) inverse of a exists.

Theorem 2.2. *Let M be a Γ-semigroup. M is an inverse Γ-semigroup if and only if (i) M is regular and (ii) if e and f be any two α-idempotents of M then $e\alpha f = f\alpha e$.*

3. Maximum idempotent separating congruence on an inverse Γ-semigroup

Theorem 3.1. *Let M be an inverse Γ-semigroup. If e be an α-idempotent and f be a β-idempotent of M then $e\alpha f$, $f\alpha e$ are β-idempotents and $e\beta f$, $f\beta e$ are α-idempotent of M.*

Proof. Let e and f be two elements of M such that e is an α-idempotent and f is a β-idempotent. We show that $e\alpha f$ is a β-idempotent. Now $e\alpha f \in M$. Since M is an inverse Γ-semigroup let $x \in V_\delta^\gamma(e\alpha f)$. Then $(e\alpha f)\delta x\gamma(e\alpha f) = e\alpha f$ and $x\gamma(e\alpha f)\delta x = x$. Let $g = f\delta x\gamma e\alpha f$. Also let $h = f\delta x\gamma e$. Then $(f\delta x\gamma e\alpha f)\beta(f\delta x\gamma e)\alpha(f\delta x\gamma e\alpha f) = f\delta(x\gamma e\alpha f\delta x)\gamma e\alpha f$ $= f\delta x\gamma e\alpha f = g$. This shows that $g\beta h\alpha g = g$. Similarly $h\alpha g\beta h = h$. Hence $g \in V_\alpha^\beta(h)$. Also, $e\alpha f \in V_\alpha^\beta(h)$. Since M is an inverse Γ-semigroup therefore $g = e\alpha f$. Hence $e\alpha f$ is β-idempotent. Proceeding similarly we can show that $f\alpha e$ is β-idempotent and $e\beta f$, $f\beta e$ are α-idempotents. Hence the theorem.

The following lemma can easily be proved.

Lemma 3.2. *Let M be an inverse Γ-semigroup. Let $a \in M$. If $a' \in V_\gamma^\delta(a)$, then for any α-idempotent e of M*

(i) *$a\gamma e\alpha a'$, $a\alpha e\gamma a'$, are δ-idempotents*

(ii) *$a'\delta e\alpha a$, $a'\alpha e\delta a$ are γ-idempotents.*

Lemma 3.3. *Let M be an inverse Γ-semigroup. Let $a, b \in M$. If $a' \in V_{\alpha_1}^{\alpha_2}(a)$, $b' \in V_{\beta_1}^{\beta_2}(b)$, then $b'\beta_2 a' \in V_{\beta_1}^{\alpha_2}(a\alpha_1 b)$, and $b'\alpha_1 a' \in V_{\beta_1}^{\alpha_2}(a\beta_2 b)$.*

Proof. Let $a' \in V_{\alpha_1}^{\alpha_2}(a)$, $b' \in V_{\beta_1}^{\beta_2}(b)$. Then $a\alpha_1 a'\alpha_2 a = a$, $a'\alpha_2 a\alpha_1 a' = a'$, $b\beta_1 b'\beta_2 b = b$, $b'\beta_2 b\beta_1 b' = b'$. Now $a'\alpha_2 a$ is α_1-idempotent and $b\beta_1 b'$ is β_2-idempotent. Hence $a'\alpha_2 a\alpha_1 b\beta_1 b'$ is β_2-idempotent, $b\beta_1 b'\beta_2 a'\alpha_2 a$ is α_1-idempotent, $a'\alpha_2 a\beta_2 b\beta_1 b'$ is α_1-idempotent and $b\beta_1 b'\alpha_1 a'\alpha_2 a$ is β_2-idempotent.

$$\begin{aligned}(a\alpha_1 b)\beta_1(b'\beta_2 a')\alpha_2(a\alpha_1 b) &= a\alpha_1(a'\alpha_2 a\alpha_1 b\beta_1 b')\beta_2(a'\alpha_2 a\alpha_1 b\beta_1 b')\beta_2 b\\ &= a\alpha_1 a'\alpha_2 a\alpha_1 b\beta_1 b'\beta_2 b\\ &\quad (\text{since } a'\alpha_2 a\alpha_1 b\beta_1 b' \text{ is } \beta_2 - \text{ idempotent})\\ &= a\alpha_1 b\end{aligned}$$

$$\begin{aligned}(b'\beta_2 a')\alpha_2(a\alpha_1 b)\beta_1(b'\beta_2 a') &= b'\beta_2(b\beta_1 b'\beta_2 a'\alpha_2 a)\alpha_1(b\beta_1 b'\beta_2 a'\alpha_2 a)\alpha_1 a'\\ &= b'\beta_2 b\beta_1 b'\beta_2 a'\alpha_2 a\alpha_1 a'\\ &\quad (\text{since } (b\beta_1 b'\beta_2 a'\alpha_2 a \text{ is } \alpha_1 - \text{ idempotent})\\ &= b'\beta_2 a'.\end{aligned}$$

Hence $b'\beta_2 a' \in V_{\beta_1}^{\alpha_2}(a\alpha_1 b)$. Similarly it can be shown that $b'\alpha_1 a' \in V_{\beta_1}^{\alpha_2}(a\beta_2 b)$.

Lemma 3.4. *Let M be a regular Γ-semigroup. Let $a, b \in M$. Then $a\mathcal{H}b$ if and only if there exists $a' \in V_\gamma^\delta(a)$, $b' \in V_\gamma^\delta(b)$ such that $a\gamma a' = b\gamma b'$, $a'\delta a = b'\delta b$.*

Proof. Let $a\mathcal{H}b$. Then $a\mathcal{L}b$ and $a\mathcal{R}b$. Let $a' \in V_\gamma^\delta(a)$. Then $a\mathcal{L}a'\delta a$ and $a\mathcal{R}a\gamma a'$. Again $a\mathcal{R}b$. Then $a\gamma a'\mathcal{R}b\mathcal{L}a'\delta a$. By Lemma 2.1, it follows that there exists a unique $b' \in V_\gamma^\delta(b)$ such that $b\gamma b' = a\gamma a'$, $b'\delta b = a'\delta a$. Conversely let $a\gamma a' = b\gamma b'$, $a'\delta a = b'\delta b$ for $a' \in V_\gamma^\delta(a)$ and $b' \in V_\gamma^\delta(b)$. Then $a\mathcal{R}a\gamma a' = b\gamma b'\mathcal{R}b$. Hence $a\mathcal{R}b$. Again $a\mathcal{L}a'\delta a = b'\delta b\mathcal{L}b$. Then $a\mathcal{L}b$. Hence $a\mathcal{H}b$.

Definition. Let M be a Γ-semigroup. A congruence on M is defined as an equivalence relation ρ on the set M stable under left and right Γ-operation. That is, for every $a, b, c \in M$, $(a, b) \in \rho$ implies $(c\alpha a, c\alpha b) \in \rho$ and $(a\alpha c, b\alpha c) \in \rho$, for all $\alpha \in \Gamma$. A left (right) congruence on M is an equivalence relation on M, stable under left(right) Γ-opeation.

Definition. Let M be a Γ-semigroup. A congruence ρ on M is said to be an idempotent separating congruence on M if e be an α-idempotent, f be an α-idempotent of M and $(e, f) \in \rho$ then $e = f$.

We can prove the following lemma.

Lemma 3.5. *Let M be a regular Γ-semigroup. If ρ is an idempotent separating congruence on M then $\rho \subset \mathcal{H}$.*

Theorem 3.6. *Let M be an inverse Γ-semigroup. Then the relation $\mu = \{(a, b) \in M \times M$: there exist $\gamma, \delta \in \Gamma$, $a' \in V_\gamma^\delta(a)$, $b' \in V_\gamma^\delta(b)$ satisfying $a\alpha e\gamma a' = b\alpha e\gamma b'$ for every α-idempotent $e = e\alpha e \in M\}$ is the maximum idempotent separating congruence on M.*

Proof. It is immediate that μ is reflexive and symmetric. Let us prove that μ is transitive. Suppose $(a, b) \in \mu$ and $(b, c) \in \mu$. Then by definition there exist $\gamma, \delta \in \Gamma$, $a' \in V_\gamma^\delta(a)$, $b' \in V_\gamma^\delta(b)$ such that $a\alpha e\gamma a' = b\alpha e\gamma b'$, for every idempotent $e = e\alpha e \in M$. From $(b, c) \in \mu$, there exist $\gamma_1, \delta_1 \in \Gamma$, $b^* \in V_{\gamma_1}^{\delta_1}(b)$, $c^* \in V_{\gamma_1}^{\delta_1}(c)$ such that $b\beta f\gamma_1 b^* = c\beta f\gamma_1 c^*$, for every idempotent $f' = f\beta f \in M$. Now

$$\begin{aligned} b\gamma b'\delta a &= b\gamma b'\delta a\gamma a'\delta a\gamma a'\delta a = b\gamma b'\delta(a\gamma(a'\delta a)\gamma a')\delta a \\ &= b\gamma b'\delta b\gamma(a'\delta a)\gamma b'\delta a = (b\gamma(a'\delta a)\gamma b')\delta a \\ &= a\gamma(a'\delta a)\gamma a'\delta a = a. \end{aligned}$$

Then $a\gamma_1(b^*\delta_1 b)\gamma a' = b\gamma_1 b^*\delta_1 b\gamma b' = b\gamma b'$. Let $\bar{a} = b^*\delta_1 b\gamma a'\delta b\gamma b'$.

$$\begin{aligned} a\gamma_1\bar{a}\delta a &= a\gamma_1 b^*\delta_1 b\gamma a'\delta b\gamma b'\delta a = (a\gamma_1(b^*\delta_1 b)\gamma a')\delta b\gamma b'\delta a \\ &= b\gamma_1(b^*\delta_1 b)\gamma b'\delta b\gamma b'\delta a = b\gamma b'\delta a = a \end{aligned}$$

$$\begin{aligned}\bar{a}\delta a\gamma_1\bar{a} &= b^*\delta_1 b\gamma a'\delta b\gamma b'\delta a\gamma_1 b^*\delta_1 b\gamma a'\delta b\gamma b' \\ &= b^*\delta_1 b\gamma a'\delta(b\gamma b'\delta a)\gamma_1 b^*\delta_1 b\gamma a'\delta b\gamma b' \\ &= b^*\delta_1 b\gamma a'\delta(a\gamma_1(b^*\delta_1 b)\gamma a')\delta b\gamma b' \\ &= b^*\delta_1 b\gamma a'\delta b\gamma_1 b^*\delta_1 b\gamma b'\delta b\gamma b' \\ &= b^*\delta_1 b\gamma a'\delta b\gamma b' = \bar{a}.\end{aligned}$$

Hence $\bar{a} \in V_{\gamma_1}^{\delta}(a)$. Next let $\bar{c} = b^*\delta_1 b\gamma_1 c^*\delta_1 b\gamma b'$.

$$\begin{aligned}b\gamma b'\delta c &= b\gamma b'\delta(c\gamma_1(c^*\delta_1 c)\gamma_1 c^*)\delta_1 c = b\gamma b'\delta b\gamma_1(c^*\delta_1 c)\gamma_1 b^*\delta_1 c \\ &= (b\gamma_1(c^*\delta_1 c)\gamma_1 b^*)\delta_1 c = c\gamma(c^*\delta_1 c)\gamma_1 c^*\delta_1 c = c.\end{aligned}$$

$$c\gamma_1(b^*\delta_1 b)\gamma_1 c^* = b\gamma_1(b^*\delta_1 b)\gamma_1 b^* = b\gamma_1 b^*.$$

Then

$$\begin{aligned}c\gamma_1\bar{c}\delta c &= (c\gamma_1 b^*\delta_1 b\gamma_1 c^*)\delta_1 b\gamma b'\delta c = b\gamma_1 b^*\delta_1 b\gamma b'\delta c \\ &= b\gamma b'\delta c = c\end{aligned}$$

$$\begin{aligned}\bar{c}\delta c\gamma_1\bar{c} &= b^*\delta_1 b\gamma_1 b^*\delta_1 b\gamma_1 c^*\delta_1(b\gamma b'\delta c)\gamma_1 c^*\delta_1 b\gamma b' \\ &= b^*\delta_1 b\gamma_1 c^*\delta_1(c\gamma_1(b^*\delta_1 b)\gamma_1 c^*)\delta_1 b\gamma b' \\ &= b^*\delta_1 b\gamma_1 c^*\delta_1 b\gamma_1 b^*\delta_1 b\gamma_1 b^*\delta_1 b\gamma b' \\ &= b^*\delta_1 b\gamma_1 c^*\delta_1 b\gamma_1 b' = \bar{c}.\end{aligned}$$

Hence $\bar{c} \in V_{\gamma_1}^{\delta}(c)$. Then

$$\begin{aligned}a\alpha e\gamma_1\bar{a} &= (a\alpha(e\gamma_1 b^*\delta_1 b)\gamma a')\delta b\gamma b' \\ &= b\alpha(e\gamma_1 b^*\delta_1 b)\gamma b'\delta b\gamma b' \\ &= c\alpha e\gamma_1\bar{c}.\end{aligned}$$

Therefore $(a, c) \in \mu$. Hence μ is transitive.

To show μ is a congruence, let $(a, b) \in \mu$ and $c \in M$, $\beta \in \Gamma$. Then by definition there exist $\gamma, \delta \in \Gamma$ and $a' \in V_{\gamma}^{\delta}(a)$, $b' \in V_{\gamma}^{\delta}(b)$ such that $a\alpha e\gamma a' = b\alpha e\gamma b'$ for every idempotent $e = e\alpha e \in M$. Now, $a' \in V_{\gamma}^{\delta}(a)$. Let $x \in V_{\gamma_2}^{\delta_2}(a'\delta a\beta c)$. Then by lemma 3.3, $x\delta_2 a' \in V_{\gamma_2}^{\delta}(a\gamma a'\delta a\beta c)$. Hence, $x\delta_2 a' \in V_{\gamma_2}^{\delta}(a\beta c)$. Similarly from $b' \in V_{\gamma}^{\delta}(b)$ and $x \in V_{\gamma_2}^{\delta_2}(a'\delta a\beta c)$ we get, $x\delta_2 b' \in V_{\gamma_2}^{\delta}(b\gamma a'\delta a\beta c)$.

$$(x\delta_2 a'\delta a\gamma b')\delta(b\beta c)\gamma_2(x\delta_2 a'\delta a\gamma b')$$

$$\begin{aligned}
&= x\delta_2(a'\delta a)\gamma(b'\delta b)\beta c\gamma_2 x\delta_2 a'\delta a\gamma b' \\
&= x\delta_2(b'\delta b)\gamma(a'\delta a)\beta c\gamma_2 x\delta_2(a'\delta a)\gamma(b'\delta b)\gamma b' \text{ (by Theorem 2.2)} \\
&= (x\delta_2 b'\delta b\gamma a'\delta a\beta c\gamma_2 x\delta_2 b')\delta b\gamma a'\delta a\gamma b' \\
&= x\delta_2 b'\delta b\gamma a'\delta a\gamma b' \text{ (since } x\delta_2 b' \in V^{\delta}_{\gamma_2}(b\gamma a'\delta a\beta c)) \\
&= x\delta_2(b'\delta b)\gamma(a'\delta a)\gamma b' = x\delta_2 a'\delta a\gamma b'\delta b\gamma b' \\
&= x\delta_2 a'\delta a\gamma b'.
\end{aligned}$$

Similarly $(b\beta c)\gamma_2(x\delta_2 a'\delta a\gamma b')\delta(b\beta c) = b\beta c$. Hence, $x\delta_2 a'\delta a\gamma b' \in V^{\delta}_{\gamma_2}(b\beta c)$. Next we show that $c\alpha e\gamma_2 x\delta_2 a'\delta a$ is β-idempotent. Now, $e\gamma_2 x\delta_2 a'\delta a\beta c$ is α-idempotent and $x\delta_2 a'\delta a\beta c$ is γ_2-idempotent.

$$\begin{aligned}
& (c\alpha e\gamma_2 x\delta_2 a'\delta a)\beta(c\alpha e\gamma_2 x\delta_2 a'\delta a) \\
&= c\alpha(e\gamma_2 x\delta_2 a'\delta a\beta c)\alpha e\gamma_2 x\delta_2 a'\delta a\beta c\gamma_2 x\delta_2 a'\delta a \\
&= c\alpha e\alpha e\gamma_2 x\delta_2 a'\delta a\beta c\gamma_2 x\delta_2 a'\delta a\beta c\gamma_2 x\delta_2 a'\delta a \text{ (by Theorem 2.2)} \\
&= c\alpha e\gamma_2(x\delta_2 a'\delta a\beta c\gamma_2 x)\delta_2 a'\delta a\beta c\gamma_2 x\delta_2 a'\delta a \\
&= c\alpha e\gamma_2(x\delta_2 a'\delta a\beta c\gamma_2 x)\delta_2 a'\delta a \\
&= c\alpha e\gamma_2 x\delta_2 a'\delta a.
\end{aligned}$$

Then,

$$\begin{aligned}
(a\beta c)\alpha e\gamma_2(x\delta_2 a') &= a\beta(c\alpha e\gamma_2 x\delta_2 a'\delta a)\gamma a' \\
&= b\beta c\alpha e\gamma_2 x\delta_2 a'\delta a\gamma b' \\
&= (b\beta c)\alpha e\gamma_2(x\delta_2 a'\delta a\gamma b').
\end{aligned}$$

This shows that $(a\beta c, b\beta c) \in \mu$. Next let $y \in V^{\delta_3}_{\gamma_3}(c\beta a\gamma a')$. Also, $a' \in V^{\delta}_{\gamma}(a)$. Then by Theorem 3.3, $a'\gamma_3 y \in V^{\delta_3}_{\gamma}(c\beta a\gamma a'\delta a)$. Thus $a'\gamma_3 y \in V^{\delta_3}_{\gamma}(c\beta a)$. Similarly from $y \in V^{\delta_3}_{\gamma_3}(c\beta a\gamma a')$ and $b' \in V^{\delta}_{\gamma}(b)$ we get, $b'\gamma_3 y \in V^{\delta_3}_{\gamma}(c\beta a\gamma a'\delta b)$. Now,

$$\begin{aligned}
a\gamma a'\delta b &= a\gamma a'\delta b\gamma b'\delta b\gamma b'\delta b = a\gamma a'\delta(b\gamma(b'\delta b)\gamma b'\delta b) \\
&= a\gamma a'\delta a\gamma(b'\delta b)\gamma a'\delta b = (a\gamma(b'\delta b)\gamma a')\delta b \\
&= b\gamma(b'\delta b)\gamma b'\delta b = b.
\end{aligned}$$

Thus $b'\gamma_3 y \in V^{\delta_3}_{\gamma}(c\beta b)$. Then,

$$(c\beta a)\alpha e\gamma(a'\gamma_3 y) = c\beta(a\alpha e\gamma a')\gamma_3 y = c\beta b\alpha e\gamma b'\gamma_3 y = (c\beta b)\alpha e\gamma(b'\gamma_3 y).$$

This implies that $(c\beta a, c\beta b) \in \mu$. Hence μ is a congruence on M. To show μ is idempotent-separating congruence on M, let e and f be two

α-idempotents of M such that $(e,f) \in \mu$. We have to show that $e = f$. Now, $(e,f) \in \mu$. Then by definition there exist $\gamma_4, \delta_4 \in \Gamma$, $e' \in V_{\gamma_4}^{\delta_4}(e)$, $f' \in V_{\gamma_4}^{\delta_4}(f)$ such that $e\nu g\gamma_4 e' = f\nu g\gamma_4 f'$, for every idempotent $g = g\nu g \in M$. Now,

$$\begin{aligned} e\alpha f &= f\alpha e = f\alpha(e\gamma_4(e'\delta_4 e)\gamma_4 e')\delta_4 e \\ &= f\alpha e\gamma_4(e'\delta_4 e)\gamma_4 f'\delta_4 e \\ &= (f\gamma_4(e'\delta_4 e)\gamma_4 f')\delta_4 e \\ &= e\gamma_4(e'\delta_4 e)\gamma_4 e'\delta_4 e = e. \end{aligned}$$

Again, $e\alpha e\gamma_4 e' = f\alpha e\gamma_4 f'$. So, $e\gamma_4 e' = f\alpha e\gamma_4 f'$. $e\alpha f\gamma_4 e' = f\alpha f\gamma_4 f'$. So, $f\gamma_4 f' = e\alpha f\gamma_4 e'$. Hence, $e\alpha f\gamma_4 f' = e\alpha f\gamma_4 e'$. Then,

$$e\alpha f = (e\alpha f\gamma_4 f')\delta_4 f = (e\alpha f\gamma_4 e')\delta_4 f = f\alpha f\gamma_4 f'\delta_4 f = f.$$

Hence $e = f$. Thus μ is idempotent separating congruence on M. Finally, suppose that ρ is an idempotent-separating congruence on M. If $(a,b) \in \rho$ then we have by Lemma 3.5 $(a,b) \in \mathcal{H}$. Then by Lemma 3.4, there exist $a' \in V_\gamma^\delta(a)$, $b' \in V_\gamma^\delta(b)$ such that $a\gamma a' = b\gamma b'$, $a'\delta a = b'\delta b$. Then $a' = a'\delta a\gamma a' = a'\delta b\gamma b'$ and $b' = b'\delta b\gamma b' = a'\delta a\gamma b'$. Since $(a,b) \in \rho, (a'\delta a, a'\delta b) \in \rho$ and accordingly $(a'\delta a\gamma b', a'\delta b\gamma b') \in \rho$. Hence $(b',a') \in \rho$. Again, $(a,b) \in \rho$ implies $(a\alpha e\gamma a', b\alpha e\gamma a') \in \rho$. Also, $(a',b') \in \rho$ implies $(b\alpha e\gamma a', b\alpha e\gamma b') \in \rho$. Hence $(a\alpha e\gamma a', b\alpha e\gamma b') \in \rho$. Then $a\alpha e\gamma a' = b\alpha e\gamma b'$, since both are δ-idempotents. Hence $\rho \subset \mu$. Thus μ is the maximum idempotent-separating congruence on M.

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