Kyungpook Mathematical Journal
Vol.34, No.1, 11-19, June 1994

# FIXED POINT THEOREMS FOR LIPSCHITZIAN SEMIGROUPS IN $p$-UNIFORMLY CONVEX BANACH SPACES*

Sung Kag Chang and Sang Sik Shin

## 1. Introduction

Let $X$ be a Banach space with norm $\| \cdot \|$ and let $C$ be a nonempty closed convex subset of $X$. A mapping $T$ of $C$ into itself is said to be Lipschitzian if there exists a positive number $k$ such that

$$\| Tx - Ty \| \leq k \| x - y \| \quad \text{for all } x, y \in C.$$

If $k$ can be taken to be 1, then the mapping $T$ is said to be nonexpansive.

Goebel and Kirk [3] first studied the existence of fixed points for uniformly Lipschitzian mappings in uniformly convex Banach space. Since then, many authors have studied fixed point properties including ergodic properties for Lipschitzian mappings and/or Lipschitzian semigroups in Hilbert spaces and uniformly convex Banach spaces. For instance, Lifschitz [6] proved that uniformly k-Lipschitzian mappings in a Hillbert space have fixed points if $k < \sqrt{2}$. Lim[7,8] studied fixed point theorems for uniformly Lipschitzian mapping in $L^p$-spaces. Lau[5] showed that a nonexpansive semigroup in a Hilbert space has a fixed point by using left invariant means. Also Downing and Ray[2] and Ishihara and Takahashi[4] showed that a uniformly k-Lipschitzian semigroup which is left reversible has a commom fixed point in a Hilbert space if $k < \sqrt{2}$. Recently Mizoguchi and Takahashi [9] showed that a Lipschitzian semigroup in a Hilbert space has a common fixed point by using a left invariant submean. Also Xu[12] studied the existence of common fixed points for a

Received April 23, 1993.
Supported by TGRC-KOSEF, 1992.

uniformly Lipschitzian semigroup in a uniformly convex space by using a left invariant mean.

In this paper, our purpose is to give a fixed point theorem for a Lipschitzian semigroup in a p-uniformly convex Banach space by using a left invariant submean.

Our main result generalizes those of Mizoguchi and Takahashi[9] and Xu[12].

## 2. Preliminaries

Throughout this paper, the Banach space $X$ is assumed to be p-uniformly convex with $p > 1$.

Then we have a lemma.

**Lemma 2.1**[12]. *If a Banach space $X$ is p-uniformly convex with $p > 1$, then there exists a constant $d > 0$ such that for all $x, y \in X$ and $0 < \lambda < 1$,*

$$\begin{aligned} &\| \lambda x + (1-\lambda)y \|^p \qquad (2.1) \\ &\leq \lambda \| x \|^p + (1-\lambda) \| y \|^p - d(\lambda(1-\lambda)^p + \lambda^p(1-\lambda)) \| x - y \|^p . \end{aligned}$$

*Remark* [13]. The constant $d$ can be taken as

$$\begin{aligned} d \;=\; & \inf\{(\lambda \| x \|^p + (1-\lambda) \| y \|^p - \| \lambda x + (1-\lambda)y \|^p)/(\lambda(1-\lambda)^p \\ & +\lambda^p(1-\lambda)) \colon 0 < \lambda < 1,\ x, y \in X \text{ with } \| x - y \| = 1\} \end{aligned}$$

Let $S$ be a semitopological semigroup. That is, $S$ is a semigroup with Hausdorff topology such that for mapping $t \mapsto st$ and $s \mapsto st$ are continuous for $s, t \in S$. Let $B(S)$ be the Banach space of all bounded real valued functions on $S$ with supremum norm $| \cdot |_\infty$, and let $B$ denote a subspace of $B(S)$ which contains all constant functions.Now we introduce a submean on $B$ which is a generalization of "mean" and "limsup".

**Definition.** A *submean* $\mu$ on $B$ is a real valued function on $B$ satisfying the following conditions:

(1) $\mu(f+g) \leq \mu(f) + \mu(g)$ for all $f, g \in B$,
(2) $\mu(\lambda f) = \lambda \mu(f)$ for all $f \in B$ and real constants $\lambda \geq 0$,
(3) $\mu(f) \leq \mu(g)$ if $f \leq g$ for $f, g \in B$
(4) $\mu(c) = c$ for all constant functions $c$.

Occasionally, we use the notation $\mu_t(f)$ instead of $\mu(f)$ in order to indicate the variable of functions $f$.

**Lemma 2.2.** *Let $C$ be a nonempty closed convex subset of $X$ and let $\{x_t \mid t \in S\}$ be a bounded set in $X$. Assume that the function $f_x(t) = \| x_t - x \|^p$ is in $B$ for each $x \in C$. Let*

$$r(x) = \mu(f_x(t)) \text{ for } x \in C.$$

*Then $r(x)$ is a continous and convex functions on $C$. Furthermore, there exists a unique $z \in C$ such that*

$$r(z) = \inf\{r(x) \mid x \in C\}$$

*and*

$$r(z) + d \| z - x \|^p \leq r(x) \text{ for all } x \in C \tag{2.2}$$

*Proof.* We show first that $r(x)$ is continuous. Let $\{x_n\}$ be a sequence in $C$ converging to $x$. Then, using the inequality

$$|a^p - b^p| \leq p|a-b|(a^{p-1} + b^{p-1}) \text{ for } a, b > 0 \text{ and } p \geq 1,$$

we have for all $n$, and $t \in S$,

$$\begin{aligned} & |\| x_t - x_n \|^p - \| x_t - x \|^p| \\ \leq\ & p\, |\| x_t - x_n \| - \| x_t - x \|| \, (\| x_t - x_n \|^{p-1} + \| x_t - x \|^{p-1}) \\ \leq\ & p \| x_n - x \| \, (\| x_t - x_n \|^{p-1} + \| x_t - x \|^{p-1}) \end{aligned}$$

Since $\{x_t\}$ and $\{x_n\}$ are bounded, $p(\| x_t - x_n \|^{p-1} + \| x_t - x \|^{p-1})$ is bounded by some number $M$ uniformly on $n$ and $t$. Thus we have for all $n$, and all $t$,

$$|\| x_t - x_n \|^p - \| x_t - x \|^p| \leq M \| x_n - x \|$$

By taking the submean $\mu$, we get

$$| r(x_n) - r(x) | \leq M \| x_n - x \| \text{ for all } n.$$

Hence $r$ is continous on $C$.

For $x, y \in C$ and $0 < \lambda < 1$, from (2.1)

$$\begin{aligned} \| x_t - \lambda x - (1-\lambda) y \|^p &= \| \lambda(x_t - x) + (1-\lambda)(x_t - y) \|^p \\ &\leq \lambda \| x_t - x \|^p + (1-\lambda) \| x_t - y \|^p \text{ for all } t \in S \end{aligned}$$

By taking the submean $\mu$, we have

$$r(\lambda x + (1-\lambda)y) \leq \lambda r(x) + (1-\lambda)r(y).$$

Which shows $r(x)$ is convex.

Since $\{x_t\}$ is bounded, $r(x_n) \to \infty$ for any sequence $\{x_n\}$ in $C$ with $\| x_n \| \to \infty$. From [1], there exists $z \in C$ such that

$$r(z) = \inf\{r(x) \mid x \in C\}$$

For any $x \in C$ and $t \in S$, $0 < \lambda < 1$, we have from (2.1)

$$\begin{aligned} &\| x_t - \lambda x - (1-\lambda)z \|^p \\ &\quad \leq \lambda \| x_t - x \|^p + (1-\lambda) \| x_t - z \|^p - d(\lambda(1-\lambda)^p + \lambda^p(1-\lambda)) \| x - z \|^p . \end{aligned}$$

By taking the submean $\mu$, we have for all $0 \leq \lambda \leq 1$,

$$r(z) \leq r(\lambda x + (1-\lambda)z) \leq \lambda r(x) + (1-\lambda)r(z) - d \| x - z \|^p .$$

Thus $r(z) + d \| x - z \|^p \leq r(x)$, which implies also the uniqueness of such minimal element $z$ in $C$.

**Definition.** Assume that $B$ is a left translation invariant subspace of $B(S)$. That is, the function $t \mapsto f(st)$ is also in $B$ for every $f \in B$ and every $s \in S$. A *submean* $\mu$ on $B$ is said to be *left invariant* if $\mu(f(t)) = \mu_t(f(st))$ for all $f \in B$ and $s \in S$.

## 3. Main Results

In this section, we give a fixed point theorem and an existence of a continuous retraction of $C$ onto the set of fixed points for a Lipschitzian semigroup.

**Definition.** A family $\Im = \{T_t \mid t \in S\}$ of mappings $T_t$ of $C$ into itself is said to be a *Lipschitzian semigroup* on $C$ if the followings are satisfied ;
(1) $T_{st}x = T_sT_tx$ for all $x \in C$, $s, t \in S$,
(2) the mapping $t \mapsto T_tx$ is continuous of $S$ into $X$ for each $x \in C$,
(3) $T_t$ is a Lipschitzian mapping on $C$ for each $t \in S$ i.e. there exists a constant $k_t \geq 0$ such that

$$\| T_tx - T_ty \| \leq k_t \| x - y \| \quad \text{for all } x, y \in C \tag{3.1}$$

**Lemma 3.1.** *Let $\{T_t \mid t \in S\}$ be a Lipschitzian semigroup on $C$ such that $\{k_t\}$ is bounded. Assume that $\{T_tx \mid t \in S\}$ is bounded for some $x \in C$*

*and the functions* $f(t) = \| T_t y - x \|^p$ *are in* $B$ *for any* $x, y \in C$. *Let* $q(x,y) = \mu_t(\| T_t y - x \|^p)$ *for* $x, y \in C$. *Then* $q(x,y)$ *is continuous and convex in* $x \in C$ *with each fixed* $y \in C$. *Moreover there exists a unique* $z \in C$ *such that*

$$q(z,y) = \inf\{q(x,y) \mid x \in C\}$$

*for each fixed* $y \in C$. *In fact,*

$$q(z,y) + d \| z - x \|^p \leq q(x,y) \text{ for all } x \in C, \tag{3.2}$$

*Proof.* It follows from lemma 2.2 by letting $x_t = T_t y$.

**Lemma 3.2**. *Under the hypothesis of lemma 3.1, assume that the function* $g(t) = k_t^p$ *is in* $B$. *Then* $q(x,y)$ *is continuous both for* $x$ *and* $y$ *in* $C$.

*Proof.* Let $\{x_n\}$ and $\{y_n\}$ be any sequences in $C$ such that $x_n \to x$ and $y_n \to y$. Then, using the inequality in Lemma 2.2., we have

$$\begin{aligned}
&|\| T_t y_m - x_n \|^p - \| T_t y - x \|^p| \\
&\quad \leq |\| T_t y_m - x_n \|^p - \| T_t y - x_n \|^p| + |\| T_t y - x_n \|^p - \| T_t y - x \|^p| \\
&\quad \leq p \| T_t y_m - T_t y \| (\| T_t y_m - x_n \|^{p-1} + \| T_t y - x_n \|^{p-1}) \\
&\qquad + p \| x_n - x \| (\| T_t y - x_n \|^p - \| T_t y - x \|^p) \\
&\quad \leq p k_t (\| T_t y_m - x_n \|^{p-1} + \| T_t y - x_n \|^{p-1}) \| y_m - y \| \\
&\qquad + p(\| T_t y - x_n \|^p - \| T_t y - x \|^p) \| y_m - y \|
\end{aligned}$$

Since $\{y_n\}, \{x_n\}$ and $\{k_t\}$ are bounded and $\{T_t\}$ is Lipschitzian, there exists a bound $M > 0$ independent of $n, m$ and $t$ such that for all $m, n$ and $t$,

$$|\| T_t y_m - x_n \|^p - \| T_t y - x \|^p| \leq M(\| y_m - y \| + \| x_n - x \|).$$

By taking the subman $\mu$, we have

$$| q(x_n, y_m) - q(x,y) | \leq M(\| y_m - y \| + \| x_n - x \|).$$

Thus $q(x,y)$ is continuous both for $x$ and $y \in C$.

Now from lemma 3.1, we can define a mapping $G$ on $C$ such that $Gy = z$ for $y \in C$ if $q(z,y) = \inf\{q(x,y) \mid x \in C\}$. Then we have from (3.2), for every $y \in C$ and all $x \in C$

$$q(Gy, y) + d \| Gy - x \|^p \leq q(x,y), \tag{3.2a}$$

**Theorem 3.3.** *Assume that $B$ is a left translation invariant subspace of $B(S)$ and $\mu$ is a left invariant submean on $B$. Under the hypothesis of lemma 3.2, if $\mu(k_t^p) < 1 + d$, then $G^n u$ converges to a fixed point of the semigroup $\Im = \{T_t \mid t \in S\}$ as $n \to \infty$ for every $u \in C$.*

*Proof.* From (3.2a) with $x = T_s G y$ for any $y \in C$, we have

$$\begin{aligned} d \parallel Gy - T_s Gy \parallel^p &\leq \mu_t(\parallel T_t y - T_s Gy \parallel^p) - \mu_t(\parallel T_t y - Gy \parallel^p) \\ &\leq (k_s^p - 1)\mu_t(\parallel T_t y - Gy \parallel^p) \end{aligned}$$

by the left invariance of $\mu$.

i.e. $\parallel Gy - T_s Gy \parallel^p \leq \frac{(k_s^p - 1)}{d} \mu_t(\parallel T_t y - Gy \parallel^p)$ for all $s \in S$. Taking the submean $\mu_s$ on both sides, we obtain by definition of $G$

$$\mu_s(\parallel T_s Gy - Gy \parallel^p) \leq \frac{[\mu_s(k_s^p) - 1]}{d} \mu_t(\parallel T_t y - y \parallel^p).$$

Let $k = \frac{[\mu_s(k_s^p) - 1]}{d}$. Then by taking $y = G^{n-1}u$ for any $u \in C$ and integer $n \geq 1$, we get,

$$\mu_t(\parallel T_t G^n u - G^n u \parallel^p) \leq k \mu_t(\parallel T_t G^{n-1} u - G^{n-1} u \parallel^p).$$

Thus by induction, for $n \geq 1$

$$\mu_t(\parallel T_t G^n u - G^n u \parallel^p) \leq k^n \mu_t(\parallel T_t u - u \parallel^p), \tag{3.3}$$

because $\mu(k_t^p) < 1 + d$, $0 \leq k < 1$.

From (3.2a) with $x = y = G^n u$, we have

$$d \parallel G^{n+1} u - G^n u \parallel^p \leq \mu_t(\parallel T_t G^n u - G^n u \parallel^p) - \mu_t(\parallel T_t G^n u - G^{n+1} u \parallel^p).$$

Hence for $n \geq 1$, from (3.3) we have

$$\parallel G^{n+1} u - G^n u \parallel^p \leq \frac{k^n}{d} \mu_t(\parallel T_t u - u \parallel^p), \tag{3.4}$$

which implies that $\{G^n u\}$ is a Cauchy sequence in $C$. Since $C$ is closed (complete), there exists $z \in C$ such that $G^n u \to z$ as $n \to \infty$. Let $\gamma = k^{\frac{1}{p}}$. Then from (3.4) we get

$$\parallel G^n u - z \parallel \leq \frac{\gamma^n}{1 - \gamma} (\frac{1}{d} q(u, u))^{\frac{1}{p}}, n \geq 1, \tag{3.5}$$

and for $s \in S$,

$$\begin{aligned} \| z - T_s z \| &\leq \| z - G^n u \| + \| G^n u - T_s G^n u \| + \| T_s G^n u - T_s z \| \\ &\leq [1 + k(s)] \| z - G^n u \| + \| G^n u - T_s G^n u \| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Which complete the proof.

Now we have some special cases. By taking $B = RUC(S)$ and $\mu$ a left invariant mean in theorem 3.3, we have

**Corollary 3.4**[12]. *Under the hypothesis of lemma 3.2, if $S$ is a left reversible semigroup and $\sup\{k_t^p \mid t \in S\} < 1 + d$, then the semigroup $\{T_t \mid t \in S\}$ has a common fixed point.*

*Remark.* When $X$ is a Hilbert space, then constant $d$ can be taken as 1. Hence by taking $X$ a Hilbert with $d = 1$ in theorem 3.3, we have the following.

**Corollary 3.5** [9]. *Under the hypothesis of lemma 3.2, if $X$ is a Hilbert space and $\mu(k_t^2) < 2$, then the semigroup $\{T_t \mid t \in S\}$ has a common fixed point.*

From theorem 3.3., we can define a mapping $P$ on $C$ as $Px = \lim_{n \to \infty} G^n x$ for $x \in C$. Let $F(\Im)$ be the set of common fixed points for the semigroup $\Im = \{T_t \mid t \in S\}$.

Then we have a lemma.

**Lemma 3.6**. *Under the hypothesis of theorem 3.3, $G$ is a continuous mapping on $C$.*

*Proof.* Let $\{y_n\}$ be any sequence in $C$ converging to $y$. From (3.2a), we have for each $n \geq 1$,

$$q(Gy_n, y_n) + d \| Gy_n - Gy \|^p \leq q(Gy, y_n), \tag{3.6}$$

Since $q(Gy, y_n)$ is bounded, $\{Gy_n\}$ is bounded. Suppose that $\{Gy_n\}$ does not converge to $Gy$ as $n \to \infty$. Then there exist a number $c > 0$ and a subsequence $\{Gy_{n_i}\}$ of $\{Gy_n\}$ such that

$$\| Gy_{n_i} - Gy \| > c \text{ for all } i.$$

Without loss of generality, we may assume $Gy_{n_i} = Gy_i$ for all $i$.

Let $A = \{Gy_n \mid n = 1, 2, \cdots\} \cup \{Gy\}$ and let $\bar{A}$ be the closure of $A$. Take $\epsilon = \frac{1}{2} dc^p > 0$.

From the fact that for some $0 < \theta < 1$,

$$\begin{aligned} &|\| T_t y_n - x \|^p - \| T_t y - x \|^p| \\ &\leq \quad p[|\| T_t y - x \| + \theta(\| T_t y_n - x \| - \| T_t y - x \|)\,|]^{p-1} k_t \| y_n - y \|, \end{aligned}$$

we see that $q(x, y_n)$ converges to $q(x, y)$ uniformly for $x$ in any bounded set as $n \to \infty$.

Since $\bar{A}$ is closed and bounded, $q(x, y_n) \to q(x, y)$ uniformly for $x \in \bar{A}$ as $n \to \infty$. Hence there exists a number $N > 0$ such that for all $x \in \bar{A}$, $| q(x, y_n) - q(x, y) | < \epsilon$ whenever $n \geq N$. i.e. $q(x, y) - \epsilon < q(x, y_n) < q(x, y) + \epsilon$ for all $x \in \bar{A}, n \geq N$.

Since $Gy, Gy_n \in \bar{A}$, for $n \geq N$, we have

$$\begin{aligned} q(Gy, y) - \epsilon &= \inf\{q(x, y) \mid x \in \bar{A}\} - \epsilon \\ &\leq \inf\{q(x, y_n) \mid x \in \bar{A}\} = q(Gy_n, y_n) \end{aligned}$$

i.e.

$$q(Gy, y) \leq q(Gy_n, y_n) + \epsilon \text{ for } n \geq N,$$

From (3.6), thus for $n \geq N$, we have

$$q(Gy, y) \leq q(Gy, y_n) - \epsilon$$

which is a contradiction to the continuity of $q$. Hence $Gy_n$ must converge to $Gy$ as $n \to \infty$. This complete the proof.

**Theorem 3.7**. *Under the hypothesis of theorem 3.3, the mapping $P$ is a continuous retraction of $C$ onto $F(\Im)$.*

*Proof.* Obviously $Pz = z$ for any $z \in F(\Im)$. Let $\{y_n\}$ be any sequence in $C$ converging to $y$ and let $D = \{y_n \mid n = 1, 2, \cdots\} \cup \{y\}$. Then $D$ is a compact set in $C$. Therefore $q$ is bounded on $D \times D$ by some number $M > 0$. i.e. $q(x, y) \leq M$ for all $x, y \in D$. From (3.5), $G^n x$ converges to $Px$ uniformly for $x \in D$. Hence for any $\epsilon > 0$, there exists a number $N > 0$ such that

$$\| G^n x - Px \| < \frac{\epsilon}{3} \text{ for all } x \in D \text{ and } n \geq N.$$

Since $G^N$ is continuous on $C$, there exists $\bar{N} > 0$ such that $\| G^N y_m - G^N y \| < \frac{\epsilon}{3}$ for all $m \geq \bar{N}$. Thus for any $m \geq \bar{N}$, we get

$$\begin{aligned} \| Py_m - Py \| &\leq \| G^N y_m - G^N y \| + \| Py_m - G^N y_m \| + \| G^N y - Py \| \\ &< \epsilon. \end{aligned}$$

This implies that $Py_m \to Py$ as $m \to \infty$.

# References


[1] V. Barbu and T. Precupanu, *Convexity and optimization in Banach spaces*, Editura Acad. R.S.R. Bucuresti, 1978.

[2] D. J. Downing and W. O. Ray, *Uniformly Lipschitzian semigroups in Hilbert space*, Canad. Math. 25, 1982, 210-214.

[3] K. Goebel and W. Kirk, *A fixed point theorem for transformations whose iterates have uniform Lipschitzian constant*, Studia Math. 47, 1973, 135-140.

[4] H. Ishihara and W. Takahashi, *Fixed point theorems for uniformly Lipschitzian semigroups in Hilbert spaces*, J.Math. Anal. Appl. 127, 1987, 206-210.

[5] A. Lau, *Semigroup of nonexpansive mappings on a Hilbert space*, J.Math. Anal. Appl. 105, 1985, 514-522.

[6] E. Lifschitz, *Fixed point theorems for operators in strongly convex spaces*, Voronez Gos. Univ. Trudy Math. Fak. 16, 1975, 23-28.

[7] T. Lim, *Fixed point theorems for uniformly Lipschitzian mappings in $L^p$ spaces*, Nonlinear Anal. 7, 1983, 555-563.

[8] T. Lim, *Some $L^p$ inequalities and their applications to fixes point theorems of uniformly Lipschitzian mappings*, Proc. Sympos. Pure Math. 45, 1986, 119-125.

[9] N. Mizoguchi and W. Takahashi, *On the existance of fixed points and ergodic retractions for Lipschitzian semigroups in Hilbert spaces*, Nonlinear Anal. 14, 1990, 69-80.

[10] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hillbert space*, Proc. Am. Math. Soc. 81, 1981, 253-256.

[11] W. Takahashi *Fixed point theorems for families of nonexpansive mappings on unbounded set*, J. Math. Soc. Japan 36, 1984, 543-553.

[12] H. Xu, *Fixed point theorems for uniformly Lipschitzian semigroups in uniformly convex spaces*, J. Math. Anal. Appl. 152, 1990, 391-398.

[13] H. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. 16, 1991, 1127-1138.

[14] H. Xu, *Existance and convergence for fixed points of mappings of asymptotically nonexpansive type*, Nonlinear Anal. 16, 1991, 1139-1146.



Department of Mathematics, Yeungnam University, Kyungsan 713-749, Korea.

Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea.