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ON GENERALIZATION OF THE CONVOLUTION THEOREM

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Efros [2] and Yadava [4] have given the generalization of the convolution theorem and evaluated the convolution type integral respectively in the recent past. In the present paper authors have further generalized the convolution theorem involving n functions of which the results proved by Efros [2] follow as a special case.

Introduction

The Laplace transform of the function $f(x)$ denoted as $F(p)$ is given by

$$F(p) = \int_0^\infty e^{-px} f(x)dx, \quad Re(p) > 0. \tag{1}$$

We shall denote (1) as

$$F(p) = L\{f(x)\}.$$

The convolution of n functions denoted as $f_1 * f_2 * \cdots * f_n(x)$ can be put in the form

$$\begin{aligned} f_1 * f_2 * \cdots * f_n(x) \;=\; & \int_0^x \int_0^{\sigma_1} \cdots \int_0^{\sigma_{n-2}} f_1(x-\sigma_1) f_2(\sigma_1-\sigma_2) \cdots \\ & f_{n-2}(\sigma_{n-3}-\sigma_{n-2}) f_{n-1}(\sigma_{n-2}-\alpha) f_n(\alpha) d\alpha d\sigma_{n-2} \cdots d\sigma_1. \end{aligned}$$

Laplace transform of this has been obtained [1, p.40] as

$$L\{f_1 * f_2 * \cdots * f_n(x)\} = F_1(p)F_2(p)\cdots F_n(p) \tag{2}$$

where $F_i(p)$ is the Laplace transform of $f_i(x), i = 1, 2, \cdots, n$. Result in (2) is the well known convolution theorem.

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Recently Efros [2] has given the generalization of the convolution theorem (2) with $n = 2$. In the present paper we have generalized the convolution theorem (2). The result proved by Efros follows as a special case of our result.

Theorem. *Let*

$$L\{f_i(x;\sigma)\} = e^{-P_i(p)\sigma} F_i(p), \cdots, \quad i = 1, 2, \cdots, n-1. \tag{3}$$

and

$$L\{f_n(x)\} = F_n(p), \tag{4}$$

then,

$$\begin{aligned} &L\{\int_0^\infty \int_0^\infty \cdots \int_0^\infty f_1(x;\sigma_1) f_2(\sigma_1;\sigma_2) \cdots f_{n-2}(\sigma_{n-3};\sigma_{n-2}) \\ &\quad \times f_{n-1}(\sigma_{n-2};\alpha) f_n(\alpha) d\alpha d\sigma_{n-2} \cdots d\sigma_1\} \\ &= F_1(p) F_2(p_1(p)) F_3(p_2(p_1(p))) \cdots F_n(p_{n-1}(p_{n-2}(\cdots(p_1(p))\cdots))), \end{aligned} \tag{5}$$

where $p_i(p), i = 1, 2, \cdots, n-1$ are the analytic functions in the region in which the Laplace transform of the functions $f_i(x, \sigma), i = 1, 2, \cdots, n$ exist. The functions $f_i(x)$ are all zero when $x < 0$. Also $f_n(x, 0) = f_n(x)$.

Proof. Consider the integral

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty f_1(x;\sigma_1) f_2(\sigma_1;\sigma_2) \cdots f_{n-2}(\sigma_{n-3};\sigma_{n-2}) \\ &\quad \times f_{n-1}(\sigma_{n-2};\alpha) f_n(\alpha) d\alpha d\sigma_{n-2} \cdots d\sigma_1. \end{aligned} \tag{6}$$

Multiplying (6) with e^{-px} and integrating the resulting expression from 0 to ∞, we get

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty f_1(x;\sigma_1) f_2(\sigma_1;\sigma_2) \cdots f_{n-2}(\sigma_{n-3};\sigma_{n-2}) \\ &\quad \times f_{n-1}(\sigma_{n-2};\alpha) f_n(\alpha) e^{-px} d\alpha d\sigma_{n-2} \cdots d\sigma_1 dx. \end{aligned} \tag{7}$$

Assuming, that the conditions, that allow the change of order of integrations are satisfied in this expression, we obtain

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty \{\int_0^\infty e^{-px} f_1(x;\sigma_1) dx\} f_2(\sigma_1;\sigma_2) \cdots f_{n-2}(\sigma_{n-3};\sigma_{n-2}) \\ &\quad \times f_{n-1}(\sigma_{n-2};\alpha) f_n(\alpha) d\sigma_1 d\sigma_2 \cdots d\sigma_{n-2} d\alpha. \end{aligned} \tag{8}$$

Making use of (3) with $i=1$, from (7) we get

$$\begin{aligned} I &= F_1(p)\int_0^\infty\int_0^\infty\cdots\int_0^\infty\{\int_0^\infty e^{-P_1(p)\sigma_1}f_2(\sigma_1;\sigma_2)d\sigma_1\}f_3(\sigma_2,\sigma_3) \\ &\quad\times\cdots f_{n-2}(\sigma_{n-3};\sigma_{n-2})f_{n-1}(\sigma_{n-2};\alpha)f_n(\alpha)d\sigma_2\cdots d\sigma_{n-2}d\alpha. \qquad (9)\end{aligned}$$

Now using (3), with $i=2$, from (8) we get

$$\begin{aligned} I &= F_1(p)F_2(p_1(p))\int_0^\infty\int_0^\infty\cdots\int_0^\infty\{\int_0^\infty e^{-P_1(P_1)\sigma_2}f_3(\sigma_2;\sigma_3)d\sigma_2\} \\ &\quad\times f_4(\sigma_3;\sigma_4)\cdots f_{n-2}(\sigma_{n-3};\sigma_{n-2})f_{n-1}(\sigma_{n-2};\alpha)f_n(\alpha)d\sigma_3\cdots d\sigma_{n-2}d\alpha.\end{aligned}$$

Proceeding this way, after n steps we get

$$\begin{aligned} I &= L\{\int_0^\infty\int_0^\infty\cdots\int_0^\infty f_1(x;\sigma_1)f_2(\sigma_1;\sigma_2)\cdots f_{n-2}(\sigma_{n-3};\sigma_{n-2}) \\ &\quad\times f_{n-1}(\sigma_{n-2};\alpha)f_n(\alpha)d\alpha d\sigma_{n-2}\cdots d\alpha_1\} \\ &= F_1(p)F_2(P_1(p))F_3(P_2(P_1(p)))\cdots F_n(P_{n-1}(P_{n-2}(\cdots(P_1(p))\cdots))),\end{aligned}$$

which is the desired result that we wanted to prove.

Particular cases

Taking $n=2$ in the theorem we get

Corollary 1. *Let* $e^{-P_1(p)\sigma}F_1(p)=L\{f_1(x;\sigma)\}$, *and* $F_2(p)=L\{f_2(x;\sigma)\}$, *then*

$$L\{\int_0^\infty f_1(x;\sigma)f_2(\sigma)d\sigma\}=F_1(p)F_2(P_1(p)) \qquad (10)$$

where $P_1(p)$ *is an analytic function of* p*, which is a know result [2].*

Further taking $P_1(p)=P_2(p)=\cdots=P_{n-1}(p)=p$ and

$$\begin{aligned} f_1(x;\sigma_1) &= f_1(x-\sigma_1), \\ f_2(\sigma_1;\sigma_2) &= f_2(\sigma_1-\sigma_2),\cdots, \\ f_{n-2}(\sigma_{n-3};\sigma_{n-2}) &= f_{n-2}(\sigma_{n-3}-\sigma_{n-2}), \\ f_{n-1}(\sigma_{n-2};\alpha) &= f_{n-1}(\sigma_{n-2}-\alpha)\end{aligned}$$

in the theorem and also noting that $f_{n-1}(\sigma_{n-2}-x)=0$ for $\alpha>\sigma_{n-2}$, $f_{n-2}(\sigma_{n-3}-\sigma_{n-2})=0$ for $\sigma_{n-2}>\sigma_{n-3},\cdots,f_2(\sigma_2-\sigma_1)=0$ for $\sigma_1>\sigma_2$ and $f_1(x-\sigma_1)=0$ for $\sigma_1>x$, from (5) we get

$$\begin{aligned} &L\{\int_0^x\int_0^{\sigma_1}\cdots\int_0^{\sigma_{n-2}} f_1(x;\sigma_1)f_2(\sigma_1-\sigma_2)\cdots f_{n-2}(\sigma_{n-3};\sigma_{n-2}) \\ &\quad\times f_{n-1}(\sigma_{n-2}-\alpha)f_n(\alpha)d\alpha d\sigma_{n-2}\cdots d\alpha_1\} \\ &= F_1(p)F_2(p)\cdots F_n(p).\end{aligned}$$

which is the well known convolution theorem (2).

References

[1] Churchill, R.V. (1958), *Operational Mathematics*, McGraw-Hill Book Company INC.

[2] Efros, A.M. (1935), *On some Applications of operator Calculus to Analysis*, Mat Sbornic, 42, 6, pp. 699-706.

[3] Shtokalo, I.X. (1976), *Operational Calculus*, Hindusten Publishing Corporation (India), Delhi.

[4] Yadava, S.R. (1971), *On a convolution type integral-II*, Bulletin of the technical Univ. of Istanbull, 24, I.

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