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SCALING d-MEASURES

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1. Introduction

In [2], the Hausdorff measures which obey a simple scaling law were investigated. Recently, d-measure was introduced to overcome difficulties in the theoretical development of a dimensional index induced by lower capacity [1]. In this note, we are interested in the characterization of continuous increasing functions θ by which d-measures d^θ obeys a scaling law. We obtain the exactly same results for d^θ as those in [2]. Thus we could get the inter-relations of Hausdorff measure, D-pre-measures and d-measures satisfying a simple scaling law.

2. Preliminaries

Let θ be a continuous increasing function defined on $\mathbf{R}^+$ with $\theta(0) = 0$. We define a pre-measure D^θ of $F \subset \mathbf{R}^m$ by $D^\theta(F) = \underline{\lim}_{r\to 0} N(F,r)\theta(r)$, where $N(F,r)$ is the minimum number of closed balls in $\mathbf{R}^m$ with diameter r, needed to cover F. Then $D^\theta(\phi) = 0$, $D^\theta(F) = D^\theta(\overline{F})$, and D^θ is monotone. We employ Method I by Munroe [3] to obtain an outer measure d^θ of $E \subset \mathbf{R}^m$; $d^\theta(E) = \inf\{\sum_{n=1}^\infty D^\theta(E_n) : \cup_{n=1}^\infty E_n = E\}$. In particular, when $\theta(t) = t^\alpha$, d^θ is the α-dimensional d-measure [1]. It is not difficult to show that d^θ is a Borel regular and metric outer measure (cf. [1]). Clearly, d^θ is a regular outer measure (cf. [1]). Also, using the subadditivity of Hausdorff outer measure $\mathcal{H}^\theta$ and the definition of d^θ, we easily see that $\mathcal{H}^\theta(E) \le d^\theta(E)$ for every set $E \subset \mathbf{R}^m$. We say that d^θ obeys an order α scaling law provided whenever $K \subset \mathbf{R}^m$ and $c > 0$, then $d^\theta(cK) = c^\alpha d^\theta(K)$.

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3. Main results

Throughout this section, we assume that θ is any continuous increasing map of $\mathbf{R}^+$ into $\mathbf{R}^+$ with $\theta(0) = 0$ such that θ is strictly concave down on a right neighborhood of 0 : there is some $\delta > 0$ such that if $0 \le x < y < \delta$ and $0 < t < 1$,

$$\theta(tx + (1-t)y) > t\theta(x) + (1-t)\theta(y) \qquad [2].$$

R.D. Mauldin and S.C. Williams construct a special Cantor set $\mathbf{C} = \cap_n [\cup_{\sigma \in \omega^*, |\sigma| = n} J_\sigma]$, induced by θ such that $0 < \mathcal{H}^\theta(\mathbf{C}) < \infty$. (See [2] for the details.)

Lemma 1. *For the special Cantor set* $\mathbf{C}$ *induced by* θ *in [2],*

$$0 < \mathcal{H}^\theta(\mathbf{C}) = d^\theta(\mathbf{C}) = D^\theta(\mathbf{C}) < \infty.$$

Proof. Since $\mathcal{H}^\theta(\mathbf{C}) \le d^\theta(\mathbf{C}) \le D^\theta(\mathbf{C})$, we only need to show that $D^\theta(\mathbf{C}) \le \mathcal{H}^\theta(\mathbf{C})$. Considering the sequence $\{m_n\}$ and $\{x_n\}$ induced by θ (cf. Lemma 6 in [2]), we have

$$\begin{aligned} D^\theta(\mathbf{C}) &\le \underline{\lim}_n N(\mathbf{C}, x_n)\theta(x_n) \\ &\le \underline{\lim}_n \prod_{i=1}^{n} m_i \theta(x_n) = \mathcal{H}^\theta(\mathbf{C}). \end{aligned}$$

Lemma 2([2]). *Suppose that for all* $c > 0$

$$\underline{\lim}_{t \to 0} \frac{\theta(ct)}{\theta(t)} = c^\alpha.$$

Then, for all $c > 0$, $\lim_{t \to 0} \frac{\theta(ct)}{\theta(t)} = c^\alpha$.

Proposition 3. *Suppose that for all* $c > 0$

$$\lim_{t \to 0} \frac{\theta(ct)}{\theta(t)} = c^\alpha.$$

If $K \subset \mathbf{R}^m$ *and* $c > 0$, *then* $D^\theta(cK) = c^\alpha D^\theta(K)$.

Proof. Noting $N(cK, cr) = N(K, r)$, we obtain the result using the similar method as the proof of Theorem 4 in [2].

Corollary 4. *Suppose that for all* $c > 0$

$$\underline{\lim}_{t\to 0}\frac{\theta(ct)}{\theta(t)} = c^{\alpha}.$$

If $K \subset \mathbf{R}^m$ *and* $c > 0$, *then* $D^{\theta}(cK) = c^{\alpha} D^{\theta}(K)$.

Proof. It follows immediately from Lemma 2 and Proposition 3.

Theorem 5. *Suppose that for all* $c > 0$ *and* $K \subset \mathbf{R}^1$, $d^{\theta}(cK) = c^{\alpha} d^{\theta}(K)$. *Then, for all* $c > 0$, $\underline{\lim}_{t\to 0}\dfrac{\theta(ct)}{\theta(t)} = c^{\alpha}$.

Proof. First, we show that $\underline{\lim}_{t\to 0}\dfrac{\theta(ct)}{\theta(t)} \le c^{\alpha}$. From Lemma 1, we assure that there exists $K \subset \mathbf{R}^1$ such that $0 < d^{\theta}(K) < \infty$. Suppose that $\underline{\lim}_{t\to 0}\dfrac{\theta(ct)}{\theta(t)} \ge Ac^{\alpha}$ for $A > B > 1$. Then there exists $\varepsilon_0 > 0$ such that $\theta(c\varepsilon) > Bc^{\alpha}\theta(\varepsilon)$ for all $0 < \varepsilon < \varepsilon_0$. Thus,

$$\begin{aligned}
d^{\theta}(cK) &= \inf\{\sum_{n=1}^{\infty} D^{\theta}(cE_n) : \cup_{n=1}^{\infty} E_n = K\} \\
&= \inf\{\sum_{n=1}^{\infty} \underline{\lim}_{\varepsilon\to 0} N(E_n,\varepsilon)\theta(c\varepsilon) : \cup_{n=1}^{\infty} E_n = K\} \\
&\ge \inf\{\sum_{n=1}^{\infty} \underline{\lim}_{\varepsilon\to 0} N(E_n,\varepsilon)Bc^{\alpha}\theta(\varepsilon) : \cup_{n=1}^{\infty} E_n = K\} \\
&= Bc^{\alpha}\inf\{\sum_{n=1}^{\infty} D^{\theta}(E_n) : \cup_{n=1}^{\infty} E_n = K\} \\
&= Bc^{\alpha} d^{\theta}(K).
\end{aligned}$$

Therefore $c^{\alpha} d^{\theta}(K) = d^{\theta}(cK) \ge Bc^{\alpha} d^{\theta}(K)$. It is a contradiction. It remains to show $\underline{\lim}_{t\to 0}\dfrac{\theta(ct)}{\theta(t)} \ge c^{\alpha}$. Fix $c > 0$ and let the sequence $\{z_n\}$ decrease to zero with

$$\lim_{n\to\infty}\frac{\theta(cz_n)}{\theta(z_n)} = \underline{\lim}_{t\to\infty}\frac{\theta(ct)}{\theta(t)}.$$

Here, we consider the special Cantor set $\mathbf{C}$ induced by the subsequence $\{x_n\}$ of $\{z_n\}$, which is constructed from θ in [2]. Then, from Lemma 1, we have

$$0 < \mathcal{H}^{\theta}(\mathbf{C}) = d^{\theta}(\mathbf{C}) < \infty.$$

Clearly

$$\begin{aligned} d^\theta(c\mathbf{C}) &\leq D^\theta(c\mathbf{C}) \\ &\leq \underline{\lim}_{n\to\infty} N(c\mathbf{C}, cx_n)\theta(cx_n) \\ &= \underline{\lim}_{n\to\infty} \prod_{i=1}^{n} m_i \theta(cx_n) \quad (\text{cf } [2]). \end{aligned}$$

Thus,

$$\begin{aligned} c^\alpha d^\theta(\mathbf{C}) &= d^\theta(c\mathbf{C}) \\ &\leq \underline{\lim}_{n\to\infty} \frac{\theta(cx_n)}{\theta(x_n)} [\theta(x_n) \prod_{i=1}^{n} m_i] \\ &= \underline{\lim}_{n\to\infty} \frac{\theta(cx_n)}{\theta(x_n)} d^\theta(\mathbf{C}). \end{aligned}$$

Hence, for each $c > 0$,

$$c^\alpha \leq \underline{\lim}_{n\to\infty} \frac{\theta(cx_n)}{\theta(x_n)} = \underline{\lim}_{t\to 0} \frac{\theta(ct)}{\theta(t)}.$$

Corollary 6. *The following five statements are equivalent.*

(i) *If* $c > 0$, *then* $\lim_{t\to 0} \frac{\theta(ct)}{\theta(t)} = c^\alpha$.

(ii) *If* $K \subset \mathbf{R}^m$ *and* $c > 0$, *then*

$$\mathcal{H}^\theta(cK) = c^\alpha \mathcal{H}^\theta(K).$$

(iii) *If* $K \subset \mathbf{R}^1$ *and* $c > 0$, *then*

$$\mathcal{H}^\theta(cK) = c^\alpha \mathcal{H}^\theta(K).$$

(iv) *If* $K \subset \mathbf{R}^1$ *and* $c > 0$, *then*

$$D^\theta(cK) = c^\alpha D^\theta(K).$$

(v) *If* $K \subset \mathbf{R}^1$ *and* $c > 0$, *then*

$$d^\theta(cK) = c^\alpha d^\theta(K).$$

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follows from Theorem 5 in [2]. It follows from Proposition 3 and Theorem 5 with Lemma 2 that i) $\Rightarrow$ iv) and v) $\Rightarrow$ i). iv) $\Rightarrow$ v) is trivial by the definition of d^θ.

References

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