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SOME CLASSES OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS I

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Abstract

Let $Q_{n+p-1}(\alpha)$ denote the class of functions $f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ $(a_{p+k} \geq 0, p \in \mathbb{N} = \{1, 2, \cdots\})$ which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$ and satisfying

$$Re\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\} > \alpha, \; 0 \leq \alpha < 1, \; n > -p, \; z \in U.$$

In this paper we obtain sharp results concerning coefficient estimates, distortion theorem, closure theorems and radii of p-valent close-to- convexity, starlikeness and convexity for the class Q_{n+p-1} (α). We also obtain class preserving integral operators of the form

$$F(z) = \frac{c+p}{z^c} \int_o^z t^{c-1} f(t) dt. \quad c > -p$$

for the class $Q_{n+p-1}(\alpha)$. Conversely when $F(z) \in Q_{n+p-1}(\alpha)$, radius of p-valence of $f(z)$ has been determined.

1. Introduction

Let $S(p)$ denote the class of functions of the form

$$(1.1) \qquad f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} (p \in \mathbb{N} = \{1, 2, \cdots, \}),$$

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which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. Let $f(z)$ be in $S(p)$ and $g(z)$ be in $S(p)$. Then we denote by $f * g(z)$ the Hadamard product of $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} (p \in \mathbf{N}), \tag{1.2}$$

then

$$f(z) * g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}. \tag{1.3}$$

The $(n+p-1)$-th order Ruscheweyh derivative $D^{n+p-1}f(z)$ of a function $f(z)$ of $S(p)$ is defined by

$$D^{n+p-1}f(z) = \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n+p-1)!} \tag{1.4}$$

where n is any integer such that $n > -p$. It is easy to see that

$$D^{n+p-1}f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) \tag{1.5}$$

$$= z^p + \sum_{k=1}^{\infty} \delta(n,k) a_{p+k} z^{p+k}. \tag{1.6}$$

where

$$\delta(n,k) = \binom{n+p-1+k}{n+p-1}. \tag{1.7}$$

Particularly, the symbol $D^n f(z)$ was named the n-th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [1].

In [3] Goel and Sohi introduced the classes $T_{n+p-1}(\alpha)$ of functions in $S(p)$ satisfying

$$Re\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\} > \alpha,\ 0 \le \alpha < 1,\ n > -p,\ z \in U. \tag{1.8}$$

Further Goel and Sohi [3] showed the basic property

$$T_{n+p}(\alpha) \subset T_{n+p-1}(\alpha) \quad (0 \le \alpha < 1,\, n > -p). \tag{1.9}$$

Let $T(p)$ denote the subclass of $S(p)$ consisting of analytic and p- valent functions which can be expressed in the form:

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} (a_{p+k} \ge 0\,;\, p \in \mathbf{N}). \tag{1.10}$$

The object of the present paper is to introduce the class $Q_{n+p-1}(\alpha)$ of analytic and p-valent functions $f(z)$ belonging to the class $T(p)$ and satisfying

$$Re\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\} > \alpha,\ 0 \le \alpha < 1,\ n > -p,\ z \in U. \tag{1.11}$$

We note that for $p = 1$ the class $Q_n(\alpha)$ ($0 \le \alpha < 1$ and $n \in \mathbf{N}_o = \mathbf{N} \cup \{0\}$) denote the subclass of $T(1)$ whose members satisfy

$$Re(D^n f(z))' > \alpha, \quad z \in U. \tag{1.12}$$

The class $Q_n(\alpha)$ was studied by Uralegaddi and Sarangi [6]. Also for $n = 1 - p$ the class $Q_0(\alpha) = F_p(1, p\alpha)$ $(0 \le \alpha < 1)$ denote the subclass of $T(p)$ whose members satisfy

$$Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} > \alpha, \quad z \in U. \tag{1.13}$$

The class $F_p(1, p\alpha)$ was studied by Lee, Owa and Srivastava [5].

Also let $T^*(p, \alpha)$ and $C(p, \alpha)$ be the subclasses of $T(p)$ that are p-valent starlike of order α and p-valent convex of order $\alpha (0 \le \alpha < p)$, respectively (see [5, 2]).

2. Coefficient Estimates

THEOREM 1. *A function $f(z)$ defined by (1.10) is in the class $Q_{n+p-1}(\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)\delta(n,k)a_{p+k} \le 1 - \alpha. \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z| = 1$. It is sufficient to show that $\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}$ lies in a circle with center at $w = 1$ and radius $1-\alpha$, we have

$$\left|\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} - 1\right| \leq \sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)a_{p+k}.$$

The last expression is bounded above by $1-\alpha$ if (2.1) is satisfied.

Conversely suppose that

$$Re\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} = Re\{1 - \sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)a_{p+k}z^k\} > \alpha;$$

choose values of z on the real axis so that $\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}$ is real. Letting $z \to 1^-$ along the real axis we obtain (2.1).

Finally, we note that the assertion (2.1) of Theorem 1 is sharp, the extremal function being

$$f(z) = z^p - \frac{p(1-\alpha)}{(p+1)\delta(n,k)}z^{p+k} \quad (k \geq 1). \tag{2.2}$$

COROLLARY 1. *Let the function $f(z)$ defined by (1.10) be in the class $Q_{n+p-1}(\alpha)$. Then we have*

$$a_{p+k} \leq \frac{p(1-\alpha)}{(p+k)\delta(n,k)} \quad (k \geq 1). \tag{2.3}$$

Equality is attained for the function $f(z)$ given by (2.2).

THEOREM 2. *$Q_{n+p}(\alpha) \subseteq Q_{n+p-1}(\alpha)$ for each $n > -p$.*

Proof. Let $f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k}z^{p+k} \in Q_{n+p}(\alpha)$; then

$$\sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n+1,k)\,a_{p+k} \leq 1-\alpha \tag{2.4}$$

and since

$$\delta(n,k) \leq \delta(n+1,k) \quad \text{for } k = 1, 2, \cdots, \tag{2.5}$$

we have

$$\sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)\,a_{p+k} \leq 1-\alpha. \tag{2.6}$$

The result follows from Theorem 1.

3. Distortion Theorem

Theorem 3. *Let the function $f(z)$ defined by (1.10) be in the class $Q_{n+p-1}(\alpha)$, $(n > -p)$, then for $|z| = r < 1$, we have*

$$r^p - \frac{p(1-\alpha)}{(p+1)(n+p)}r^{p+1} \leq |f(z)| \leq r^p + \frac{p(1-\alpha)}{(p+1)(n+p)}r^{p+1}, \tag{3.1}$$

and

$$pr^{p-1} - \frac{p(1-\alpha)}{(n+p)}r^p \leq |f'(z)| \leq pr^{p-1} + \frac{p(1-\alpha)}{(n+p)}r^p. \tag{3.2}$$

Furthermore

$$p - p(1-\alpha)r \leq \left|\frac{(D^{n+p-1}f(z))'}{z^{p-1}}\right| \leq p + p(1-\alpha)r. \tag{3.3}$$

These results are sharp.

Proof. Since $f(z) \in Q_{n+p-1}(\alpha)$, in view of Theorem 1, we have

$$\begin{aligned}(\frac{p+1}{p})\delta(n,1)\sum_{k=1}^{\infty} a_{p+k} &\leq \sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)a_{p+k} \\ &\leq (1-\alpha),\end{aligned} \tag{3.4}$$

which evidently yields

$$\sum_{k=1}^{\infty} a_{p+k} \leq \frac{p(1-\alpha)}{(p+1)(n+p)} \quad (n > -p). \tag{3.5}$$

Consequently, we obtain

$$\begin{aligned}|f(z)| &\geq r^p - r^{p+1}\sum_{k=1}^{\infty} a_{p+k} \\ &\geq r^p - \frac{p(1-\alpha)}{(p+1)(n+p)}r^{p+1} \quad (n > -p),\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}|f(z)| &\leq r^p + r^{p+1}\sum_{k=1}^{\infty} a_{p+k} \\ &\leq r^p + \frac{p(1-\alpha)}{(p+1)(n+p)}r^{p+1} \quad (n > -p),\end{aligned} \tag{3.7}$$

which prove the assertion (3.1) of Theorem 3. Further

$$|f'(z)| \geq pr^{p-1} - r^p \sum_{k=1}^{\infty}(p+k)a_{p+k}$$

and

$$|f'(z)| \leq pr^{p-1} + r^p \sum_{k=1}^{\infty}(p+k)a_{p+k}.$$

But from Theorem 1, it holds that

$$\frac{\delta(n,1)}{p}\sum_{k=1}^{\infty}(p+k)a_{p+k} \leq \sum_{k=1}^{\infty}\frac{(p+k)}{p}\delta(n,k)a_{p+k} \leq (1-\alpha)$$

which gives that

$$\sum_{k=1}^{\infty}(p+k)a_{p+k} \leq \frac{p(1-\alpha)}{(n+p)} \quad (n > -p). \tag{3.8}$$

Hence

$$|f'(z)| \geq pr^{p-1} - \frac{p(1-\alpha)}{(n+p)}r^p \quad (n > -p) \tag{3.9}$$

and

$$|f'(z)| \leq pr^{p-1} + \frac{p(1-\alpha)}{(n+p)}r^p \quad (n > -p) \tag{3.10}$$

which prove the assertion (3.2) of Theorem 3.

Next, by using the second inequality in (3.4), we observe that

$$\begin{aligned} \left|\frac{(D^{n+p-1}f(z))'}{z^{p-1}}\right| &\leq p + r\sum_{k=1}^{\infty}(p+k)\delta(n,k)a_{p+k} \\ &\leq p + p(1-\alpha)r, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \left|\frac{(D^{n+p-1}f(z))'}{z^{p-1}}\right| &\geq p - r\sum_{k=1}^{\infty}(p+k)\delta(n,k)a_{p+k} \\ &\geq p - p(1-\alpha)r, \end{aligned} \tag{3.12}$$

which prove the assertion (3.3) of Theorem 3. Sharpness follows if we take

$$f(z) = z^p - \frac{p(1-\alpha)}{(p+1)(n+p)}z^{p+1} \quad (n > -p,\ z = \pm r). \tag{3.13}$$

COROLLARY 2. *Under the hypotheses of Theorem 3, $f(z)$ is included in a disc with its center at the origin and radius r_1 given by*

$$r_1 = 1 + \frac{p(1-\alpha)}{(p+1)(n+p)} \quad (n > -p), \tag{3.14}$$

and $f'(z)$ is included in a disc with its center at the origin and radius r_2 given by

$$r_2 = p + \frac{p(1-\alpha)}{(p+1)(n+p)} \quad (n > -p). \tag{3.15}$$

Also $\frac{(D^{n+p-1}f(z))'}{z^{p-1}}$ is included in a disc with its center at the origin and radius r_3 given by

$$r_3 = p + p(1-\alpha). \tag{3.16}$$

The result is sharp with extremal function $f(z)$ given by (3.13).

4. Integral Operators

THEOREM 4. *Let the function $f(z)$ defined by (1.10) be in the class $Q_{n+p-1}(\alpha)$, and let $F(z)$ be defined by*

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt. \tag{4.1}$$

Then

(i) for every c, $c > -p$, $F(z) \in Q_{n+p-1}(\alpha)$
and
(ii) for every c, $-p < c \leq n$, $F(z) \in Q_{n+p}(\alpha)$.

Proof. (i) From the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{k=1}^{\infty} b_{p+k} z^{p+k},$$

where

$$b_{p+k} = (\frac{c+p}{c+p+k})a_{p+k}.$$

Therefore

$$\sum_{k=1}^{\infty}\frac{(p+k)}{p}\delta(n,k)b_{p+k}=\sum_{k=1}^{\infty}\frac{(p+k)}{p}\delta(n,k)(\frac{c+p}{c+p+k})a_{p+k}$$
$$=\sum_{k=1}^{\infty}\frac{(p+k)}{p}\delta(n,k)a_{p+k}\leq(1-\alpha),$$

since $f(z)\in Q_{n+p-1}(\alpha)$. Hence, by Theorem 1, $F(z)\in Q_{n+p-1}(\alpha)$.

(ii) In view of Theorem 1 it is sufficient to show that

$$\sum_{k=1}^{\infty}\frac{(p+k)}{p}\delta(n+1,k)(\frac{c+p}{c+p+k})a_{p+k}\leq(1-\alpha).$$

Since $\delta(n,k)-(\frac{c+p}{c+p+k})\delta(n+1,k)\geq 0$ if $-p<c\leq n$ $(k=1,2,\cdots)$ the result follows from Theorem 1.

Putting $c=1-p$ in Theorem 4 we get the following

COROLLARY 3. *Let the function $f(z)$ defined by (1.10) be in the class $Q_{n+p-1}(\alpha)$, and let $F(z)$ be defined by*

$$F(z)=\frac{1}{z^{1-p}}\int_0^z\frac{f(t)}{t^p}dt. \tag{4.2}$$

Then $F(z)\in Q_{n+p}(\alpha)$.

THEOREM 5. *Let c be a real number such that $c>-p$. If $F(z)\in Q_{n+p-1}(\alpha)$, then the function $f(z)$ defined in (4.1) is p-valent in $|z|<R_p^*$, where*

$$R_p^*=\inf_k\left[\frac{(c+p)\delta(n,k)p}{(c+p+k)(1-\alpha)}\right]^{\frac{1}{k}}\quad(k\geq 1). \tag{4.3}$$

The result is sharp.

Proof. Let $F(z)=z^p-\sum_{k=1}^{\infty}a_{p+k}z^{p+k}(a_{p+k}\geq 0)$. It follows from (4.1) that

$$f(z)=\frac{z^{1-c}(z^cF(z))'}{(c+p)},\quad(c>-p)$$
$$=z^p-\sum_{k=1}^{\infty}(\frac{c+p+k}{c+p})a_{p+k}z^{p+k}.$$

To prove the result it suffices to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \le p \text{ for } |z| < R_p^*.$$

Now

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|-\sum_{k=1}^{\infty}(\frac{c+p+k}{c+p})a_{p+k}z^k\right|$$
$$\le \sum_{k=1}^{\infty}(\frac{c+p+k}{c+p})a_{p+k}|z|^k.$$

Thus $\left|\frac{f'(z)}{z^{p-1}} - p\right| \le p$ if

(4.4) $$\sum_{k=1}^{\infty}(\frac{c+p+k}{c+p})(p+k)a_{p+k}|z|^k \le p.$$

But Theorem 1 confirms that

(4.5) $$\sum_{k=1}^{\infty}\frac{(p+k)\delta(n,k)}{(1-\alpha)}a_{p+k} \le p.$$

Thus (4.4) will be satisfied if

(4.6) $$(\frac{c+p+k}{c+p})(p+k) \le \frac{(p+k)\delta(n,k)}{(1-\alpha)}, \quad (k \ge 1),$$

or if

$$|z| \le \left[\frac{(c+p)\delta(n,k)}{(c+p+k)(1-\alpha)}\right]^{\frac{1}{k}}, \quad (k \ge 1).$$

The required result follows now from (4.6). The result is sharp for the function

(4.7) $$f(z) = z^p - \frac{p(c+p+k)(1-\alpha)}{(p+k)(c+p)\delta(n,k)}z^{p+k} \quad (k \ge 1).$$

THEOREM 6. *Let the function $F(z)$ defined by (1.10) be in the class $Q_{n+p-1}(\alpha)$, $f(z) = \frac{z^{1-c}(z^c F(z))'}{(c+p)}$, $(c > -p)$. Then $Re\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} > \beta$ $(0 \leq \beta < 1)$ for $|z| < r_p^*$, where*

$$r_p^* = \inf_k \left[\frac{(c+p)(1-\beta)}{(c+p+k)(1-\alpha)}\right]^{\frac{1}{k}} \quad (k \geq 1). \tag{4.8}$$

The result is sharp.

Proof. It is sufficient to show that

$$\left|\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} - 1\right| \leq 1 - \beta \text{ for } |z| < r_p^*. \tag{4.9}$$

We have

$$\left|\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} - 1\right| \leq \sum_{k=1}^{\infty}(\frac{p+k}{p})(\frac{c+p+k}{c+p})\delta(n,k)a_{p+k}|z|^k.$$

Hence the inequality (4.9) will be satisfied if

$$\sum_{k=1}^{\infty}(\frac{p+k}{p})(\frac{c+p+k}{c+p})\delta(n,k)a_{p+k}|z|^k \leq 1 - \beta. \tag{4.10}$$

Since $F(z) \in Q_{n+p-1}(\alpha)$, from Theorem 1,

$$\sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)a_{p+k} \leq 1 - \alpha$$

and the inequality (4.10) will be satisfied if

$$\frac{(p+k)(c+p+k)\delta(n,k)a_{p+k}|z|^k}{p(c+p)(1-\beta)} \leq \frac{(p+k)\delta(n,k)a_{p+k}}{p(1-\alpha)}.$$

Solving it for $|z|$ we obtain

$$|z| \leq \left[\frac{(c+p)(1-\beta)}{(c+p+k)(1-\alpha)}\right]^{\frac{1}{k}} \quad \text{for} \quad k = 1, 2, \cdots.$$

Writing $|z| = r_p^*$ the result follows. The estimate is sharp for the function

$$F(z) = z^p - \frac{p(1-\alpha)}{(p+k)\delta(n,k)}z^{p+k} \text{ for som } k.$$

THEOREM 7. *Let the function $f(z)$ defined by (1.10) be in the class $T^*(p,\alpha)$ and let*

$$(4.11)\qquad F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt = z^p - \sum_{k=1}^{\infty}\left(\frac{c+p}{c+p+k}\right)a_{p+k}z^{p+k}.$$

Then

(i) *$F(z) \in T^*(p,\alpha)$ for $c > -p$*

and

(ii) *$F(z) \in C(p,\alpha)$ for $-p < c \leq 0$.*

Proof. (i) $f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k}z^{p+k} \in T^*(p,\alpha)$ if and only if

$$\sum_{k=1}^{\infty}(p+k-\alpha)a_{p+k} \leq (p-\alpha) \qquad [4,2].$$

For $c > -p$,

$$\sum_{k=1}^{\infty}\left(\frac{c+p}{c+p+k}\right)(p+k-\alpha)a_{p+k} \leq \sum_{k=1}^{\infty}(p+k-\alpha)a_{p+k} \leq (p-\alpha).$$

Hence $F(z) \in T^*(p,\alpha)$.

(ii) Let $-p < c \leq 0$. From (4. 11) we obtain

$$zF'(z) = (c+p)f(z) - cF(z).$$

Since $T^*(p,\alpha)$ is closed under convex linear combinations [4.2], $zF'(z) \in T^*(p,\alpha)$. That is $F(z) \in C(p,\alpha)$.

5. Radii of Close-to-Convexity, Starlikeness and Convexity

THEOREM 8. *Let the function $f(z)$ defined by (1.10) be in the class $Q_{n+p-1}(\alpha)$, then $f(z)$ is p-valent close-to-convex of order $\beta(0 \leq \beta < p)$ in $|z| < r_1(n,p,\alpha,\beta)$, where*

$$(5.1)\qquad r_1(n,p,\alpha,\beta) = \inf_k\left[\frac{(p-\beta)\delta(n,k)}{p(1-\alpha)}\right]^{\frac{1}{k}} \quad (k \geq 1).$$

The result is sharp, with the extremal function $f(z)$ given by (2.2).

Proof. We must show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| \le \rho - \beta$ for $|z| < r_1(n,p,\alpha,\beta)$. We have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \le \sum_{k=1}^{\infty}(p+k)a_{p+k}|z|^k.$$

Thus $\left|\frac{f'(z)}{z^{p-1}} - p\right| \le p - \beta$ if

$$\sum_{k=1}^{\infty}(\frac{p+k}{p-\beta})a_{p+k}|z|^k \le 1. \tag{5.2}$$

Hence, by Theorem 1, (5.2) will be true if

$$(\frac{p+k}{p-\beta})|z|^k \le \frac{(p+k)\delta(n,k)}{p(1-\alpha)}$$

or if

$$|z| \le \left[\frac{(p-\beta)\delta(n,k)}{p(1-\alpha)}\right]^{\frac{1}{k}} \quad (k \ge 1). \tag{5.3}$$

The theorem follows easily from (5.3).

THEOREM 9. *Let the function $f(z)$ defined by (1.10) be in the class $Q_{n+p-1}(\alpha)$, then $f(z)$ is p-valent starlike of order $\beta (0 \le \beta < p)$ in $|z| < r_2(n,p,\alpha,\beta)$, where*

$$r_2(n,p,\alpha,\beta) = \inf_k \left[\frac{(p-\beta)(p+k)\delta(n,k)}{(p+k-\beta)p(1-\alpha)}\right]^{\frac{1}{k}} \quad (k \ge 1). \tag{5.4}$$

The result is sharp, with the extremal function $f(z)$ given by (2.2).

Proof. It is sufficient to show that $\left| \frac{zf'(z)}{f(z)} - p \right| \le p - \beta$ for $|z| < r_2(n,p,\alpha,\beta)$. We have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le \frac{\sum_{k=1}^{\infty} k a_{p+k}|z|^k}{1 - \sum_{k=1}^{\infty} a_{p+k}|z|^k}$$

Thus $\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \beta$ if

$$\sum_{k=1}^{\infty} \frac{(p+k-\beta)}{(p-\beta)} a_{p+k}|z|^k \leq 1. \tag{5.5}$$

Hence, by Theorem 1, (5.5) will be true if

$$\frac{(p+k-\beta)}{(p-\beta)}|z|^k \leq \frac{(p+k)\delta(n,k)}{p(1-\alpha)}$$

or if

$$|z| \leq \left[\frac{(p-\beta)(p+k)\delta(n,k)}{(p+k-\beta)p(1-\alpha)}\right]^{\frac{1}{k}} \quad (k \geq 1). \tag{5.6}$$

The theorem follows easily from (5.6).

COROLLARY 4. *Let the function $f(z)$ defined by (1.10) be in the class $Q_{n+p-1}(\alpha)$, then $f(z)$ is p-valent convex of order $\beta(0 \leq \beta < p)$ in $|z| < r_3(n,p,\alpha,\beta)$, where*

$$r_3(n,p,\alpha,\beta) = \inf_k \left[\frac{(p-\beta)\delta(n,k)}{(p+k-\beta)(1-\alpha)}\right]^{\frac{1}{k}} \quad (k \geq 1). \tag{5.7}$$

The result is sharp, with the extremal function $f(z)$ given by (2.2)

6. Closure Theorems

Let the function $f_i(z)$ be defined, for $i = 1, 2, \cdots, m$, by

$$f_i(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,i} z^{p+k} (a_{p+k,i} \geq 0; p \in \mathbf{N}) \tag{6.1}$$

for $z \in U$.

THEOREM 10. *Let the functions $f_i(z)$ $(i = 1, 2, \cdots, m)$ defined by (6.1) be in the class $Q_{n+p-1}(\alpha)$. Then the function $h(z)$ defined by*

$$h(z) = z^p - \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \tag{6.2}$$

also belongs to the class $Q_{n+p-1}(\alpha)$, *where*

$$b_{p+k} = \frac{1}{m}\sum_{i=1}^{m} a_{p+k,i}. \tag{6.3}$$

Proof. Since $f_i(z) \in Q_{n+p-1}(\alpha)$, it follows from Theorem 1, that

$$\sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)a_{p+k,i} \leq 1-\alpha, \quad i = 1,2,\cdots,m.$$

Therefore

$$\sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)b_{p+k} = \sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)\left\{\frac{1}{m}\sum_{i=1}^{m}a_{p+k,i}\right\} \leq 1-\alpha. \tag{6.4}$$

Hence by Theorem 1, $h(z) \in Q_{n+p-1}(\alpha)$. Thus we have the theorem.

THEOREM 11. *Let the functions* $f_i(z)$ *defined by (6.1) be in the classes* $Q_{n+p-1}(\alpha_i)$ *for each* $i = 1,2,\cdots,m$. *Then the function* $h(z)$ *defined by*

$$h(z) = z^p - \frac{1}{m}\sum_{k=1}^{\infty}\left(\sum_{i=1}^{m}a_{p+k,i}\right)z^{p+k} \tag{6.5}$$

is in the class $Q_{n+p-1}(\alpha)$, *where*

$$\alpha = \min_{1\leq i\leq m}\{\alpha_i\}. \tag{6.6}$$

Proof. Since $f_i(z) \in Q_{n+p-1}(\alpha_i)$ for each $i = 1,2,\cdots,m$, we observe that

$$\sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)a_{p+k,i} \leq 1-\alpha_i \tag{6.7}$$

with the aid of Theorem 1. Therefore

$$\begin{aligned}\sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)\left(\frac{1}{m}\sum_{i=1}^{m}a_{p+k,i}\right) &= \frac{1}{m}\sum_{i=1}^{m}\left\{\sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)a_{p+k,i}\right\} \\ &\leq \frac{1}{m}\sum_{i=1}^{m}(1-\alpha_i) \leq 1-\alpha.\end{aligned}$$

Thus

$$\sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)\left(\frac{1}{m}\sum_{i=1}^{m}a_{p+k,i}\right)\leq 1-\alpha, \tag{6.8}$$

which shows that $h(z)\in Q_{n+p-1}(\alpha)$, where α is given by (6.6).

THEOREM 12. *Let the functions $f_i(z)$ defined by (6.1) be in the classes $Q_{n+p-1}(\alpha)$ for every $i=1,2,\cdots,m$. Then the function $h(z)$ defined by*

$$h(z)=\sum_{i=1}^{m}c_i f_i(z) \quad (c_i\geq 0) \tag{6.9}$$

is also in the same class $Q_{n+p-1}(\alpha)$, where

$$\sum_{i=1}^{m}c_i=1. \tag{6.10}$$

Proof. According to the definition of $h(z)$, we can write that

$$h(z)=z^p-\sum_{k=1}^{\infty}\left(\sum_{i=1}^{m}c_i a_{p+k,i}\right)z^{p+k}. \tag{6.11}$$

By means of Theorem 1, we have

$$\sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)a_{p+k,i}\leq 1-\alpha \tag{6.12}$$

for every $i=1,2,\cdots,m$. Hence we can observe that

$$\begin{aligned}&\sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)\left(\sum_{i=1}^{m}c_i a_{p+k,i}\right)\\ &=\sum_{i=1}^{n}c_i\left\{\sum_{k=1}^{\infty}(\frac{p+k}{p})\delta(n,k)a_{p+k,i}\right\}\\ &\leq\left(\sum_{i=1}^{m}c_i\right)(1-\alpha)=1-\alpha\end{aligned} \tag{6.13}$$

which implies that $h(z)\in Q_{n+p-1}(\alpha)$. Thus we have the theorem.

THEOREM 13. *THe class $Q_{n+p-1}(\alpha)$ is convex.*

Proof. Let the functions $f_i(z)$ $(i = 1, 2)$ defined by (6.1) be in the class $Q_{n+p-1}(\alpha)$. Thus it is sufficient to prove that the function

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \quad (0 \le \lambda \le 1) \tag{6.14}$$

is in the class $Q_{n+p-1}(\alpha)$. Since for $0 \le \lambda \le 1$,

$$h(z) = z^p - \sum_{k=1}^{\infty} \{\lambda a_{p+k,1} + (1-\lambda) a_{p+k,2}\} z^{p+k}, \tag{6.15}$$

with the aid of Theorem 1, we have

$$\sum_{k=1}^{\infty} (\frac{p+k}{p}) \delta(n,k) \{\lambda a_{p+k,1} + (1-\lambda) a_{p+k,2}\} \le 1-\alpha \tag{6.16}$$

which implies that $h(z) \in Q_{n+p-1}(\alpha)$. Hence $Q_{n+p-1}(\alpha)$ is convex.

THEOREM 14. *Let $f_p(z) = z^p$ and*

$$f_{p+k}(z) = z^p - \frac{p(1-\alpha)}{(p+k)\delta(n,k)} z^{p+k} \; (k \ge 1, n > -p), \tag{6.17}$$

for $0 \le \alpha < 1$. Then $f(z) \in Q_{n+p-1}(\alpha)$ if and only if it can be expressed in the form

$$f(z) = \lambda_p f_p(z) + \sum_{k=1}^{\infty} \lambda_{p+k} f_{p+k}(z), \tag{6.18}$$

where $\lambda_{p+k} \ge 0$ $(k \ge 1)$ and $\lambda_p + \sum_{k=1}^{\infty} \lambda_{p+k} = 1$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \lambda_p f_p(z) + \sum_{k=1}^{\infty} \lambda_{p+k} f_{p+k}(z) \\ &= z^p - \sum_{k=1}^{\infty} \frac{p(1-\alpha)\lambda_{p+k}}{(p+k)\delta(n,k)} z^{p+k} \end{aligned} \tag{6.19}$$

Then we get

$$
(6.20) \qquad \begin{aligned} &\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)\delta(n,k)\frac{p(1-\alpha)}{(p+k)\delta(n,k)}\lambda_{p+k} \\ &= (1-\alpha)\sum_{k=1}^{\infty}\lambda_{p+k} \leq 1-\alpha. \end{aligned}
$$

Hence, by Theorem 1, $f(z) \in Q_{n+p-1}(\alpha)$.

On the other hand, suppose that the function $f(z)$ defined by (1.10) is in the class $Q_{n+p-1}(\alpha)$. Again, by using Theorem 1, we can show that

$$
(6.21) \qquad a_{p+k} \leq \frac{p(1-\alpha)}{(p+k)\delta(n,k)} \quad (k \geq 1, n > -p).
$$

Setting

$$
(6.22) \qquad \lambda_{p+k} = \frac{(p+k)\delta(n,k)}{p(1-\alpha)} \quad (k \geq 1, n > -p).
$$

and

$$
(6.23) \qquad \lambda_p = 1 - \sum_{k=1}^{\infty}\lambda_{p+k}.
$$

Hence, we can see that $f(z)$ can be expressed in the form (6.18). This completes the proof of Theorem 14.

COROLLARY 5. *The extreme points of the class* $Q_{n+p-1}(\alpha)$ *are* $f_p(z) = z^p$ *and*

$$
f_{p+k}(z) = z^p - \frac{p(1-\alpha)}{(p+k)\delta(n,k)}z^{p+k} \quad (k \geq 1, n > -p).
$$

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