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A FUNCTIONAL CENTRAL LIMIT THEOREM FOR POSITIVELY DEPENDENT SEQUENCES

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Abstract

In this note we prove a functional central limit theorem for LPQD sequences, statisfying some moment conditions. No stationarity is required. Our results imply an extension of Birkel's functional central limit theorem for associated processes to an LPQD sequence and an improvement of Birkel's functional central limit theorem for LPQD sequences.

1. Introduction and Main results

A sequence $\{X_j : j \in N\}$ of random variables is said to be pairwise positive quadrant dependent(PQD) if for any real r_i, r_j and $i \neq j$

$$P\{X_i > r_i, X_j > r_j\} \geq P\{X_i > r_i\}P\{X_j > r_j\}.$$

This notion of positive dependence was introduced by Lehmann [5] and a much stronger concept than PQD was considered by Esary, Proschan, and Walkup [4]. A sequence $\{X_j : j \in N\}$ of random variables is said to be associated if for any finite collection $\{X_{j(1)}, \cdots, X_{j(n)}\}$ and any real coordinatewise nondecreasing functions f, g on R^n

$$\mathrm{Cov}(f(X_{j(1)}, \cdot, X_{j(n)}), g(X_{j(1)}, \cdot, X_{j(n)})) \geq 0,$$

whenever the covariance is defined. Newman [6] was the first who showed the central limit theorem for associated sequences. In the following years several extensions and generalizations of this result were given. There exist several (functional) central limit theorems [2,6,7] for associated processes.

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Most of these results, however, cannot be applied to weaker concepts of positive dependence. Instead of association Newman's original central limit theorem requires only that positively linear combinations of the random variables are PQD. The following definition of positive dependence was introduced by Newman [6] as an extension of the bivariate notion of PQD of Lehmann [5]. A sequence $\{X_j : j \in N\}$ of random variables is said to be linearly positive quadrant dependent (LPQD) if for any disjoint A, B and positive real r_j's

$$\sum_{j\in A} r_j X_j \quad \text{and} \quad \sum_{j\in B} r_j X_j \quad \text{are} \quad PQD.$$

Let us remark that this concept of dependence is between pairwise PQD and association. Using the coefficient of maximal covariances

$$u(n) = \sup_{k\in N} \sum_{j=|j-k|\geq n} \mathrm{Cov}(X_j, X_k)$$

Birkel [2] proved the central limit theorem for nonstationary associated random variables(see Theorem 3 of [2]) and it was not explicitly mentioned, but it is easy to see that this theorem still holds for LPQD processes(cf.[3]).

THEOREM A (BIRKEL). *Let $\{X_j : j \in N\}$ be an LPQD sequence with $EX_j = 0$,*
$EX_j^2 < \infty$. Assume

$$u(n) \longrightarrow_n 0, \; u(1) < \infty, \tag{1.1}$$

$$\sigma_n^{-2} \sum_{j=1}^{n} E({X_j}^2 1_{\{|X_j|\geq \epsilon\sigma_n\}}) \longrightarrow_n 0 \; \text{for } \epsilon > 0, \tag{1.2}$$

$$\inf_{n\in N} n^{-1}\sigma_n^2 > 0. \tag{1.3}$$

Then $\{X_j : j \in N\}$ fulfills the central limit theorem.
Birkel [3] also proved the following functional central limit theorem under more sharpened conditions than (1.1) and (1.2).

THEOREM B (BIRKEL). *Let $\{X_j : j \in N\}$ be an LPQD sequence with $EX_j = 0$,*
$EX_j^2 < \infty$. Assume that (1.3) and the following conditions (1.4), (1.5), and (1.6) hold.

$$(1.4) \qquad u(n) = O(n^{-\rho}) \text{ for some } \rho > 0,$$

$$(1.5) \qquad \sup_{j \in N} E|X_j|^{2+\delta} < \infty \text{ for some } \delta > 0,$$

$$(1.6) \qquad \sigma_n^{-2}\sigma_{nk}^2 \longrightarrow_n k \text{ for } k \in N.$$

Then $\{X_j : j \in N\}$ fulfills the functional central limit, that is, $W_n(t) = \sigma_n^{-1} S_{[nt]}, \quad t \in [0,1]$ converges weakly to standard Brownian motion W on the set of all functions on $[0,1]$ which have left hand limits and are continuous from the right.
In this paper we will weaken (1.4) and (1.5) to (1.1) and (1.2) and thus improve Theorem B. (Theorem of Birkel[3]).

THEOREM 1.1. *Let $\{X_j : j \in N\}$ be an LPQD sequence with $EX_j = 0, EX_j^2 < \infty$. Assume that (1.1), (1.2), (1.3), and (1.6) hold. Then $\{X_j : j \in N\}$ fulfills the functional central limit theorem.*

COROLLARY 1.2. *Let $\{X_j : j \in N\}$ be an LPQD sequence with $EX_j = 0, EX_j^2 < \infty$. Assume that (1.1), (1.2), and the following condition (1.7) hold.*

$$(1.7) \qquad n^{-1}\sigma_n^2 \longrightarrow_n \sigma^2 \ , \ 0 < \sigma^2 < \infty,$$

where

$$\sigma^2 = cov(X_1, X_1) + 2\sum_{j=2}^{\infty} cov(X_1, X_j).$$

Then $\{X_j : j \in N\}$ fulfills the functional central limit theorem.

proof. Since (1.7) implies (1.3) and (1.6) $\{X_j : j \in N\}$ fulfills the functional central limit theorem according to Theorem 1.1.
If $\{X_j : j \in N\}$ is stationary in the wide sense, $0 < \sigma_2 < \infty$ obviously implies (1.1) and (1.7). Hence we obtain:

COROLLARY 1.3. *$\{X_j : j \in N\}$ be a wide sense stationary LPQD sequence with $EX_j = 0, EX_j^2 < \infty$. Assume that $0 < \sigma_2 < \infty$ and (1.2) hold. Then $\{X_j : j \in N\}$ fulfills the functional central limit theorem.*

2. Proof

The following lemmas will be used to provide the tightness needed for our functional central limit theorem.

LEMMA 2.1. *Let $\{X_j : j \in N\}$ be an LPQD sequence with $EX_j = 0, EX_j^2 < \infty$. Define for $n \in N, m \in N \cup \{0\}, S_n = X_1 + \cdots + X_n,\ S_{m,n} = S_{n+m} - S_m,\ M_{m,n} = \max(S_{m,1}, \cdots, S_{m,n})$, Then,*

$$E(M_{m,n}^2) \leq E(S_{m,n}^2) \tag{2.1}$$

proof. This theorem can be proved along the lines of the proof of Theorem 2 of Newman and Wright [9].
We next define for $n \in N, m \in N \cup \{0\}$,

$$S_{m,n}^* = \max(0, S_{m,1}, S_{m,2}, \cdots, S_{m,n}),\ s_{m,n}^2 = E(S_{m,n}^2).$$

LEMMA 2.2. *Let $\{X_j : j \in N\}$ be an LPQD sequence with $EX_j = 0, EX_j^2 < \infty$ Then, for $\lambda_2 > \lambda_1 > 0$,*

$$P(S_{m,n}^* \geq \lambda_2) \leq (1 - s_{m,n}^2/(\lambda_2 - \lambda_1)^2)^{-1} P(S_{m,n} \geq \lambda_1) \tag{2.2}$$

$$P(\max_{1 \leq j \leq n}(|S_{m,j}|) \geq \lambda s_{m,n}) \leq 2P(|S_{m,n}| \geq (\lambda - \sqrt{2}) s_{m,n}) \tag{2.3}$$

Proof. For $\lambda_1 < \lambda_2$,
(2.4)

$$\begin{aligned} &P(S_{m,n}^* \geq \lambda_2) \\ &\leq P(S_{m,n} \geq \lambda_1) + P(S_{m,n-1}^* \geq \lambda_2, S_{m,n-1}^* - S_{m,n} > \lambda_2 - \lambda_1) \\ &\leq P(S_{m,n} \geq \lambda_1) + P(S_{m,n-1}^* \geq \lambda_2) P(S_{m,n-1}^* - S_{m,n} \geq \lambda_2 - \lambda_1) \\ &\leq P(S_{m,n} \geq \lambda_1) + P(S_{m,n}^* \geq \lambda_2) E((S_{m,n-1}^* - S_{m,n})^2)/(\lambda_2 - \lambda_1)^2 \end{aligned}$$

where the second inequality follows from the fact that $S_{m,n-1}^*$ and $S_{m,n} - S_{m,n-1}^*$ are PQD since the $X_j's$ are LPQD random variables and the third inequality follows from the Chebyshev's inequality. Now Lemma 2.1 X_{i+m} replaced by $Y_{i+m} = -X_{n-i+1+m}$ yields that,

$$\begin{aligned} &E([S^*_{m,n-1} - S_{m,n}]^2) \\ &= E([\max(Y_{1+m}, Y_{1+m} + Y_{2+m}, \cdots, Y_{1+m} + Y_{2+m} + \cdots + Y_{n+m})]^2) \\ &\leq E(S^2_{m,n}) = s^2_{m,n} \end{aligned}$$

which together with (2.4) yields (2.2) for $(\lambda_2 - \lambda_1)^2 \geq s^2_{m,n}$. By adding to (2.2) the analogous inequality with each X_{i+m} replaced by $-X_{i+m}$ in (2.2) and by choosing $\lambda_2 = \lambda s_{m,n}, \lambda_1 = (\lambda - \sqrt{2})s_{m,n}$, (2.3) will be obtained.

Proof of Theorem 1.1. : It is easy to see that Lemmas 1 and 2 of Birkel[2] still hold for random variable, which are nonnegatively correlated. Hence it follows from (1.6) that

$$\sigma_n^{-2}\sigma^2_{[nt]} \longrightarrow_n t \text{ for } t > 0, \tag{2.4}$$

$$\sigma_n^{-2}E((S_{[nt]} - S_{[ns]})(S_{[nv]} - S_{[nu]})) \longrightarrow_n 0 \text{ for } 0 \leq s \leq t \leq u \leq v. \tag{2.5}$$

Let X be a limit in distribution of a subsequence of $\{W_n : n \in N\}$. We apply some techniques on lines of the proof of Theorem of Birkel [3]. It suffices to show that X is distributed like W. By Theorem A and (2.4) we obtain for $t \in [0,1]$

$$W_n(t) \longrightarrow_n N(0,t) \qquad \text{in distribution.} \tag{2.6}$$

Hence the sets $\{W_n(t) : n \in N\}$ and $\{W_n^2(t) : n \in N\}$ are uniformly integrable. As $W_n(t) \longrightarrow_n X(t)$, $W_n^2(t) \longrightarrow_n X^2(t)$ in distribution (for a subsequence),

$$EX(t) = 0, \quad EX^2(t) = t$$

by Theorem 5.4 of Billingsely[1] and (2.4). According to Theorem 19.1 of Billingsely[1], X is distributed like W if X has independent increments, that is,

$$X(t_1) - X(t_0), \cdots, X(t_k) - X(t_{k-1}) \tag{2.7}$$

are independent for all $k \geq 1$, $0 \leq t_0 < t_1 \leq \cdots \leq t_k \leq 1$.

To show (2.7), put

$$U_{n,i} = W_n(t_i) - W_n(t_{i-1}), \quad 1 \le i \le k.$$

Then the U_{ni} are LPQD random variables, and

$$(U_{n1}, \cdots, U_{nk}) \longrightarrow_n (X(t_1) - X(t_0), \cdots, X(t_k) - X(t_{k-1}))$$

in distribution (for a subsequence). Thus by Lemma 4 of Birkel[3] the $X(t_i) - X(t_{i-1})$ are LPQD. Using Theorem 5.4 of Billingsely [1] and (2.5), we get, for $i \ne j$,

$$\mathrm{Cov}(X(t_i) - X(t_{i-1}), X(t_j) - X(t_{j-1})) = \lim_{n \in N} \mathrm{Cov}(U_{ni}, U_{nj}) = 0.$$

Hence the $X(t_i) - X(t_{j-1})$ are uncorrelated, LPQD random variables and thus independent by Theorem 6 of Newman [6]. This proves (2.7). Now it remains to prove the needed tightness. Applying (2.3) to the random variables involved in Theorem 1.1, we have for $\lambda > 2\sqrt{2}$,

$$(2.8) \qquad P\{\max_{i \le n} |S_{i+m} - S_m| \ge \lambda s_{m,n}\} \le 2P\{|S_{n+m} - S_m| \ge \frac{1}{2}\lambda s_{m,n}\}.$$

We will prove, for $0 < s < t$,

$$(2.9) \qquad {\sigma_n}^{-1}(S_{[nt]} - S_{[ns]}) \longrightarrow_n N(t-s) \text{ in distribution}$$

To show (2.9) we use the technique of the proof of Theorem 2 of Birkel[2]. Let $0 < s < t$ be given. Then the sequence

$$\{({\sigma_n}^{-1} S_{[ns]}, {\sigma_n}^{-1} S_{[nt]}) : n \in N\}$$

is tight(see [1] p 41 problem 6). Let Q be a probability measure on the Borel σ−algebra of R^2 such that for a subsequence

$$({\sigma_n}^{-1} S_{[ns]}, {\sigma_n}^{-1} S_{[nt]}) \longrightarrow_n Q \text{ in distribution.}$$

Then we have

$$({\sigma_n}^{-1} S_{[ns]}, {\sigma_n}^{-1}(S_{[nt]} - S_{[ns]})) \longrightarrow_n Q(\pi_1, \pi_2, -\pi_1)^{-1} \text{ in distribution,}$$

where $\pi_i : R^2 \to R, i = 1, 2$, are the natural projections. Since the random variables ${\sigma_n}^{-1} S_{[ns]}$ and ${\sigma_n}^{-1}(S_{[nt]} - S_{[ns]})$ are PQD, Lemma 4 of [3] implies that π_1 and $\pi_2 - \pi_1$ are PQD with respect to Q. According to (2.6), the sets $\{{\sigma_n}^{-1} S_{[ns]} : n \in N\}$, $\{{\sigma_n}^{-1} S_{[nt]} : n \in N\}$ and $\{{\sigma_n}^{-2} S_{[ns]} S_{[nt]} : n \in N\}$ are uniformly integerable. Hence, using Theorem 5.4 of Billingsley [1] and (2.5), we obtain

$$\mathrm{Cov}(\pi_1, \pi_2 - \pi_1) = \lim_{n \in N} \mathrm{Cov}({\sigma_n}^{-1} S_{[ns]}, {\sigma_n}^{-1}(S_{[nt]} - S_{[ns]})) = 0.$$

As LPQD and uncorrelated random variables, π_1 and $\pi_2 - \pi_1$ are Q-independent. Since $Q{\pi_1}^{-1} = N(0, s)$, $Q{\pi_2}^{-1} = N(0, t)$, this proves (2.9). (2.8) and (2.9) yield the needed tightness of the sequence $\{W_n : n \in N\}$ by the standard argument (see the proof of Theorem 10.1 in Billingsely[1]). Thus the proof of our theorem is complete.

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