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APPROXIMATION SCHEME FOR A CONTROL SYSTEM

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Abstract

Piezoceramic patches as collocated actuator and sensors are widely used in mechanical control systems. An approximation scheme for computing feedback gains arising in heat flux stabilization problem with such control mechanism is introduced. The scheme is based on a finite element method and a variational approach.

1. Introduction

Piezoceramic patches as control actuator and sensors are widely used in regulating mechanical systems (see, e.g., Balakrishnan [2]). In this paper, we introduce an approximation scheme for computing feedback gains for control system with unbounded observation. These feedback gains produce control signals for stabilizing a given dynamical system. We consider a linear regulator problem for stabilizing heat flux with a certain degree of decay rate. Optimal control signals for stabilizing heat flux with a certain desired exponential decay rate, say $\alpha > 0$, are often obtained from "α-shifted" control systems (see, Anderson and Moore [1], and Gibson and Rosen [5]). An α-shifted heat flux control system can be written as

(1.1)

$$\frac{\partial}{\partial t}v(t,x) = \kappa\frac{\partial^2}{\partial x^2}v(t,x) + \alpha\, v(t,x) + \sum_{i=1}^{m} b_i(x)u_i(t), \quad 0 < x < \ell,\ t > 0,$$

$$v(0,x) = v_0(x), \quad 0 \le x \le \ell,$$

$$v(t,0) = v(t,\ell) = 0, \quad t > 0,$$

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where $\kappa > 0$ is the heat conductivity, $\alpha > 0$ is a desired exponential decay rate, $\ell > 0$ is the length of the domain, b_i, $1 \le i \le m$, is the step function of the form

$$b_i(x) = \begin{cases} 1, & \bar{x}_i - d_i \le x \le \bar{x}_i + d_i, \\ 0, & \text{otherwise}, \end{cases} \tag{1.2}$$

$\bar{x}_i$ is the center of each piezoceramic patch, $2d_i$ is the length of the patch, $u_i(t)$ is the control signal acting on each $b_i(x)$ which is to be determined, and $[\bar{x}_i - d_i\ \bar{x}_i + d_i] \subset [0, \ell]$, $1 \le i \le m$. The unbounded observation operator $\mathcal{C}$ considered in this paper is defined by

$$\mathcal{C}v(\cdot) = (c_1(v), c_2(v), \cdots, c_m(v)) \in \mathbf{R}^m, \tag{1.3}$$

where $c_i(v) = \int_{\bar{x}_i - d_i}^{\bar{x}_i + d_i} v'(x)dx$, $1 \le i \le m$.

By standard semigroup techniques (see, e.g., Pazy [7]), the system (1.1)-(1.3) can be written on the state space $H = L^2(0, \ell)$ as

$$\begin{aligned} &\frac{d}{dt} z(t) = (\kappa\mathcal{A} + \alpha I)z(t) + \mathcal{B}u(t), \quad z(0) = z_0, \\ &y(t) = \mathcal{C}z(t), \end{aligned} \tag{1.4}$$

where $z(t) = z(t)(\cdot) = v(t, \cdot) \in H$, the operators $\mathcal{A}$, $\mathcal{B}$ are defined by

$$\mathcal{A}\phi = \phi'', \quad \mathcal{B}u = \sum_{i=1}^{m} b_i u_i \tag{1.5}$$

for all $\phi \in dom(\mathcal{A}) = H^2(0, \ell) \cap H_0^1(0, \ell)$. Here, $\mathcal{B} \in \mathcal{L}(U, H)$, $b_i \in H$, $1 \le i \le m$, $U = \mathbf{R}^m$, $u = (u_1, \cdots, u_m) \in U$, $\mathcal{C} \in \mathcal{L}(W, Y)$ is defined by equation (1.3), $W = H_0^1(0, \ell)$, $Y = \mathbf{R}^m$. U is the control space, $\mathcal{C}$ is the observation operator, and Y is the ouput space.

2. Linear Regulator Problem

Consider the following linear quadratic regulator problem:
(LQR): *Find the optimal control* $\bar{u}(\cdot) \in L^2(0, \infty; U)$ *that minimizes the quadratic cost functional*

$$J(u) = \int_0^\infty \left(\|y(t)\|_Y^2 + \|u(t)\|_U^2\right) dt \tag{2.1}$$

subject to the control system (1.4).

Consider the following eigenpairs of the operator $\kappa\mathcal{A} + \alpha I$:

$$\lambda_{\alpha,n} = \alpha - \frac{\kappa n^2 \pi^2}{\ell^2} \quad \text{and} \quad \phi_{\alpha,n}(x) = \sqrt{\frac{2}{\ell}} \sin \frac{n\pi}{\ell} x. \tag{2.2}$$

For small $\kappa > 0$, if $\alpha > \alpha_0 = \frac{\kappa\pi^2}{\ell^2}$, then there is at least one positive eigenvalue. Without loss of generality, we assume that $\alpha > \alpha_0$ and let

$$n_\alpha = \max\{n \in N : \lambda_{\alpha,n} = \alpha - n^2\alpha_0 \geq 0\}. \tag{2.3}$$

We then have the following theorem. Since the proof is an application of Burns and Kang [4], and Pritchard and Salamon [8], we only sketch the proof.

THEOREM 1. *Suppose that, for each $n = 1, 2, \cdots, n_\alpha$, there is at least one i, $1 \leq i \leq m$, such that*

$$\bar{x}_i, d_i \notin X = \{\frac{k\ell}{2n} : k = 0, 1, 2, \cdots, 2n-1\}, \tag{2.4}$$

where the parameters n_α, $\bar{x}_i$, d_i, $1 \leq i \leq m$, are given in equations (1.2) and (2.3). Then the linear regulator problem (LQR) has a unique optimal control $\bar{u}(\cdot) \in L^2(0, \infty; U)$ which is given by

$$\bar{u}(t) = -\mathcal{B}^*\Pi\bar{z}(t), \quad t \geq 0, \tag{2.5}$$

where $\bar{z}(t)$ is the corresponding optimal trajectory of the closed-loop control system (1.4) and $\Pi \in \mathcal{L}(H)$ is the unique nonnegative self-adjoint operator satisfying the Riccati equation

$$(\kappa\mathcal{A} + \alpha I)\Pi z + \Pi(\kappa\mathcal{A} + \alpha I)z - \Pi\mathcal{B}\mathcal{B}^*\Pi z + \mathcal{C}^*\mathcal{C}z = 0 \tag{2.6}$$

for all $z \in dom(\mathcal{A})$, where the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are defined by equations (1.4)-(1.5), $H = L^2(0, \ell)$, and $U = \mathbf{R}^m$. Moreover, the closed-loop semigroup decays exponentially.

Sketch of the proof. Since $\kappa\mathcal{A} + \alpha I$ is self-adjoint on the state space H, the set of all eigenfunctions $\phi_{\alpha,n}$ forms a basis for H, and, hence, each element $z \in H$ can be identified with the sequence $\{< z, \phi_{\alpha,n} >_H\}_{n\in N}$ and $W = H_0^1(0, \ell)$ can be represented by

$$W = \{z \in H : \sum_{n=1}^{\infty} n^2 | < z, \phi_{\alpha,n} >_H |^2 < \infty\}, \tag{2.7}$$

where $\phi_{\alpha,n}$ are given by equation (2.2). Also, the operators $\mathcal{B}$ and $\mathcal{C}$ can be represented by

(2.8) $$\mathcal{B}u = \{< b_n, u >_U\}_{n\in N} \quad \text{and} \quad \mathcal{C}z = \sum_{n=1}^{\infty} c_n < z, \phi_{\alpha,n} >_H,$$

where

(2.9)
$$b_n = (b_{n1}, b_{n2}, \cdots, b_{nm}) \in U = \mathbf{R}^m, \quad b_{ni} = \frac{2\sqrt{2\ell}}{n\pi} \sin\frac{n\pi\bar{x}_i}{\ell} \sin\frac{n\pi d_i}{\ell},$$
$$c_n = (c_{n1}, c_{n2}, \cdots, c_{nm}) \in Y = \mathbf{R}^m, \quad c_{ni} = 2\sqrt{\frac{2}{\ell}} \cos\frac{n\pi\bar{x}_i}{\ell} \sin\frac{n\pi d_i}{\ell},$$
$$(1 \le i \le m).$$

From equations (2.8) and (2.9), it is easy to see that

$$\sum_{n=1}^{\infty} \|b_n\|_U^2 \le \sum_{n=1}^{\infty} \left(\frac{8\ell m}{\pi^2}\right) \frac{1}{n^2} < \infty, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \|c_n\|_Y^2 \le \sum_{n=1}^{\infty} \left(\frac{8m}{\ell}\right) \frac{1}{n^2} < \infty.$$

Hence, $(\kappa\mathcal{A} + \alpha I, \mathcal{B})$ is stabilizable if and only if $b_n \ne 0$ for all $n = 1, 2, \cdots, n_\alpha$ if and only if for each $n = 1, 2, \cdots, n_\alpha$, there exists at least one i, $1 \le i \le m$, such that $\bar{x}_i, d_i \ne \frac{k\ell}{n}$, $k = 1, 2, \cdots, n-1$. Also, $(\kappa\mathcal{A} + \alpha I, \mathcal{C})$ is detectable if and only if $c_n \ne 0$ for all $n = 1, 2, \cdots, n_\alpha$ if and only if for each $n = 1, 2, \cdots, n_\alpha$, there exists at least one i, $1 \le i \le m$, such that $\bar{x}_i \ne \frac{(2k+1)\ell}{2n}$, $k = 0, 1, 2, \cdots, n-1$, $d_i \ne \frac{j\ell}{n}$, $j = 0, 1, 2, \cdots, n-1$. Under the assumption (2.4), both the stabilizability and detectability conditions are satisfied. Therefore, by the standard arguments of the linear regulator theory with unbounded operators, the theorem holds.

REMARK. The conditions (2.4) are sufficient for the existence of the optimal control $\bar{u}(t)$. Necessary and sufficient conditions for the stabilizability and detectability of the control system (1.4) are given in the proof of Theorem 1.

EXAMPLE. Let $\ell = 1$, $\kappa = 0.001$, $m = 1$, and $\alpha = 0.1$. Then $n_\alpha = 3$. Therefore, if the center $\bar{x}$ of the piezoceramic patch is not located at $\frac{1}{6}, \cdots, \frac{5}{6}$, and its length $2d$ is less than $\frac{1}{3}$, then the control system (1.1) can be exponentially stabilizable by the optimal feedback control $\bar{u}(t)$ given by equation (2.5).

3. Approximation scheme

In this section we consider a numerical approximation scheme for computing the feedback gain operator $\mathcal{K} = \mathcal{B}^*\Pi$ in equation (2.5), where the operator Π satisfies the Riccati equation (2.6). Numerical experiments where the observation operator $\mathcal{C}$ is bounded was reported in Burns and Kang [3]. For the numerical results with similar unbounded observation action as considered in this paper were reported in Burns and Kang [4]. For nonlinear control syntheses connected with a nonlinear system, see Ito and Kang [6].

Throughout this section, we assume the length ℓ of the domain is 1. For our finite dimensional approximation scheme, divide the unit interval [0,1] into $N+1$ equal subintervals $[x_i, x_{i+1}]$, $x_i = \frac{i}{N+1}$, $i = 0, 1, \cdots, N$. For each i, $1 \le i \le N$, define the basis functions $h_i^N(x)$ by

$$h_i^N(x) = \begin{cases} (N+1)(x - x_{i-1}), & x_{i-1} \le x \le x_i \\ -(N+1)(x - x_{i+1}), & x_i \le x \le x_{i+1} \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Let H^N denote the N-dimensional finite element space given by

$$H^N = \{ \sum_{i=1}^{N} z_i h_i^N(x) \ : \ z_i \in \mathbf{R}, \, i = 1, \cdots, N \} \tag{3.2}$$

and the approximate solution $z^N(t,x)$ of $z(t,x)$ on H^N be given by

$$z^N(t,x) = \sum_{i=1}^{N} z_i^N(t) h_i^N(x) \tag{3.3}$$

for some $z_i^N(t) \in \mathbf{R}$, $i = 1, \cdots, N$.

On H^N, the control system (1.4) can be approximated by the following finite dimensional ODE system

$$\begin{aligned} &\frac{d}{dt}\{z^N(t)\} = [-\kappa \mathcal{A}^N + \alpha I^N]\{z^N(t)\} + [\mathcal{B}^N]u^N(t), \\ &\{z^N(0)\} = \{z_0^N\}, \\ &\{y^N(t)\} = [\mathcal{C}^N]\{z^N(t)\}, \end{aligned} \tag{3.4}$$

where $\{z^N(t)\} = [z_1^N(t), \cdots, z_N^N(t)]^T$, $\{u^N(t)\} = [u_1^N(t), \cdots, u_m^N(t)]^T$,

$$[\mathcal{A}^N] = [G^N]^{-1}[\tilde{A}^N], \quad [\mathcal{B}^N] = [G^N]^{-1}[\tilde{B}]^N, \quad [\mathcal{C}^N] = [c_{ij}^N]_{m\times N},$$
$$[G^N] = [g_{ij}^N]_{N\times N}, \quad [\tilde{A}^N] = [\tilde{a}_{ij}^N]_{N\times N}, \quad [\tilde{B}^N] = [\tilde{b}_{ij}]_{N\times m},$$
$$g_{ij}^N = \begin{cases} \frac{4}{6(N+1)}, & i = j, \\ \frac{1}{6(N+1)}, & |i-j| = 1, \\ 0, & \text{otherwise}, \end{cases} \quad \tilde{a}_{ij}^N = \begin{cases} -2(N+1), & i = j, \\ (N+1), & |i-j| = 1, \\ 0, & \text{otherwise}, \end{cases}$$
for $1 \le i \le N$, $1 \le j \le N$,
$$\tilde{b}_{ij} = \int_{\bar{x}_j - d_j}^{\bar{x}_j + d_j} h_i^N(x)dx, \quad c_{ji}^N = \int_{\bar{x}_j - d_j}^{\bar{x}_j + d_j} (h_i^N)'(x)dx,$$
for $1 \le i \le N$, $1 \le j \le m$,

and $\{z_0^N\} = [G^N]^{-1}[\int_0^1 z_0(x)h_1^N(x)dx, \cdots, \int_0^1 z_0(x)h_N^N(x)dx]^T$. Hence, the corresponding finite dimensional Riccati equation for equation (2.6) is given by

(3.5)
$$(-\kappa\mathcal{A}^N + \alpha I^N)\Pi^N + \Pi^N(-\kappa\mathcal{A}^N + \alpha I^N) - \Pi^N\mathcal{B}^N(\mathcal{B}^N)^*\Pi + (\mathcal{C}^N)^*\mathcal{C}^N = 0.$$

In general, equation (3.5) has 2^N solutions. Among them, there is a unique nonnegative self-adjoint solution. To find the unique solution, standard solution methods for the Riccati equation such as Potter's method (see, e.g., Russell [9]) can be employed. The following is a brief sketch of Potter's method. The first step is to form $2N \times 2N$ matrix

$$(3.6) \qquad [M^N] = \begin{bmatrix} [-\kappa\mathcal{A}^N + \alpha I^N]^* & [\mathcal{C}^N]^*[\mathcal{C}^N] \\ [\mathcal{B}^N][\mathcal{B}^N]^* & -[-\kappa\mathcal{A}^N + \alpha I^N] \end{bmatrix}.$$

Next, find all eigenvalues and eigenvectors of M^N and form the matrix $[Z^N] = [Q_1^N\, Q_2^N]^T$, where the columns of $[Z^N]$ are eigenvectors of $[M_N]$ corresponding to the eigenvalues with positive real part. When eigenvalues occur in complex conjugate pairs, so do the eigenvectors. In this case, the real and imaginary part of the eigenvector each forms a column of $[Z^N]$. Finally, the solution to the Riccati equation (3.5) is given by the formula $\Pi^N = (Q_1^N)(Q_2^N)^{-1}$.

The finite dimensional feedback operator $[\mathcal{K}^N]$ given by

$$(3.7) \qquad [\mathcal{K}^N] = -[\mathcal{B}^N]^*[\Pi^N].$$

Using this feedback gain operator, the finite dimensional closed-loop control system becomes

$$
(3.8) \qquad \begin{aligned} &\frac{d}{dt}\{z^N(t)\} = ([-\kappa \mathcal{A}^N + \alpha I^N] - [\mathcal{B}^N][\mathcal{K}^N])\{z^N(t)\} \\ &\{z_0^N(0)\} = \{z_0^N\}. \end{aligned}
$$

The solution of the closed-loop system (3.8) is exponentially stable. Hence, by using the optimal control $u^N(t) = -\mathcal{K}^N \bar{z}^N(t)$, the unshifted control system (i.e., $\alpha = 0$ in equation (1.1)) can be stabilized with exponential decay $\alpha + \epsilon$ for some $\epsilon > 0$, where $\bar{z}^N(t)$ is the corresponding closed-loop trajectory.

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