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GENERATING FUNCTION AND SEMI–ORTHOGONAL RELATIONS FOR A CLASS OF HYPERGEOMETRIC FUNCTIONS

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Abstract

We present a generating function and three semi-orthogonal relations for a class of hypergeometric functions. We employ semi-orthogonal relations to generate a theory concerning finite series expansions involving our hypergeometric functions.

1. Introduction

The object of this paper is to present a generating function and three semi-orthogonal relations for a class of hypergeometric polynomials:

$$(1.1)\qquad {}_2F_1\left(\begin{matrix}-m; 1-b-m; \frac{x-1}{x}\\ 1-a-m\end{matrix}\right) = \sum_{n=0}^{m} \frac{(-m)_n(1-b-m)_n}{(1-a-m)_n n!}\left(\frac{x-1}{x}\right)^n,\ m=0,1,\cdots$$

We further apply semi-orthogonal relations to develop a theory regarding the finite series expansions involving our hypergeometric functions.

The Jacobi polynomials constitute an important, and a rather wide class of hypergeometric polynomials, from which Chebyshev, Legendre, Laguerre and Gegenbauer polynomials follow as special cases. Their orthogonality, with the non-negative weight function $(1-x)^a(1+x)^b$ on the interval $[-1,1]$, for $a>-1$, $b>-1$, is usually derived by the use of the associated differential equation and Rodrigues' formula. In this paper, we

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introduce a direct method of proof, which is much simpler and elegant to establish orthogonal and semi-orthogonal relations of the hypergeometric polynomials. Our method could be employed to establish the orthogonality properties of the Jacobi polynomials and other hypergeometric polynomials.

The following three integrals are required in the proofs.

(i) First Integral

$$(1.2)\quad \int_0^1 x^{h+n-1}(1-x)^{b-a-h-1}\,{}_2F_1\left[\begin{matrix}-n, 1-b-n; \frac{x-1}{x}\\ 1-a-n\end{matrix}\right]dx$$
$$= \frac{(1+h-b-n)_n\Gamma(h)\Gamma(b-a-h)}{(1-a-n)_n\Gamma(b-a)},\ n=0,1,2,\cdots$$

where $Re h > 0$, $Re(h-b) > -1$, $Re(b-a-h) > 0$.

Proof. The integral (1.2) is established by expressing the hypergeometric functions in the integrand as its series representation (1.1), interchanging the order of integration and summation, evaluating the resulting integral with the help of the Beta integral [3, p.9]:

$$(1.3)\qquad \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},\ p>0,\ q>0$$

and simplifying, with the help of the following form of the formula [2, p.3].

$$(1.4)\qquad \Gamma(1+a-n) = \frac{(-1)^n\Gamma(1+a)}{(-a)_n}$$

we get

$$(1.5)\qquad = \frac{\Gamma(h+n)\Gamma(b-a-h)}{\Gamma(b-a+n)}\,{}_3F_2\left[\begin{matrix}-n,\ 1-b-n,\ b-a-h; 1\\ 1-a-n,\ 1-h-n\end{matrix}\right].$$

It can easily be verified that the generalized hypergeometric series (1.5) is Saalschutzian. Therefore, applying the Saalschutz's theorem [3, p.188, (3)]:

$$(1.6)\qquad {}_3F_2\left[\begin{matrix}-n,\ a,\ b\ \ ; 1\\ c,\ 1-c+a+b-n\end{matrix}\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}$$

and simplifying with the help of (1.4), the right hand side of (1.2) is obtained.

(ii) Second Integral

$$(1.7)\quad \int_1^\infty x^{h+n-1}(x-1)^{b-a-h-1}\,{}_2F_1\left(\begin{matrix}-n, 1-b-n; \\ 1-a-n\end{matrix}\ \frac{x-1}{x}\right)dx$$
$$= \frac{(1+h-b-n)_n\Gamma(1+a-b)\Gamma(b-a-h)}{(1-a-n)_n\Gamma(1-h)},\ n=0,1,2,\cdots$$

where $Re(a-b)>-1$, $Re(b-a-h)>0$, $Re(h-b)>-1$.

Proof. The integral (1.7) is established on following the technique employed to establish (1.2) except instead of (1.3), using the integral [4, p.201,(7)]:

$$(1.8)\qquad \int_1^\infty x^{-v}(x-1)^{w-1}\,dx = \frac{\Gamma(v-w)\Gamma(w)}{\Gamma(v)},\ Re v > Re w > 0.$$

(iii) Third Integral

$$(1.9)\quad \int_0^\infty x^{h+n-1}(x+1)^{b-a-h-1}\,{}_2F_1\left(\begin{matrix}-n, 1-b-n; \\ 1-a-n\end{matrix}\ \frac{x+1}{x}\right)dx$$
$$= (-1)^n\frac{(1+h-b-n)_n\Gamma(1+a-b)\Gamma(h)}{(1-a-n)_n\Gamma(1+a-b+h)},\ n=0,1,2,\cdots$$

where $Re h>0$, $Re(a-b)>-1$, $Re(h-b)>-1$.

Proof. The integral (1.9) is established on following the technique employed to establish (1.2) except instead of (1.3), using the integral [4, p.233,(8)]:

$$(1.10)\qquad \int_0^\infty x^{v-1}(1+x)^{-w}\,dx = \frac{\Gamma(v)\Gamma(w-v)}{\Gamma(w)},\ Re w > Re v > 0.$$

2. Generating Function

We define the hypergeometric polynomials (1.1) by means of the generating function:

$$(2.1)\qquad e^{-t(x-1)/x}M(a;b;t)=\sum_{n=0}^{\infty}\frac{t^n(a)_n}{n!(b)_n}\,{}_2F_1\left[\begin{matrix}-n,\,1-b-n;\,\dfrac{x-1}{x}\\ 1-a-n\end{matrix}\right]$$

Proof. In a result earlier obtained by Bajpai [2, p.17, (3.3)], putting $A=B=1$, $p=q=1$, $r=s=0$, $b_1=b$, $a_1=a$ and setting $(\frac{1-x}{x})t$ for y and t for x, we have

$$(2.2)\qquad {}_1F_1(a;b;t)\,{}_0F_0(-;-;\frac{1-x}{x}t)=\sum_{m=0}^{\infty}\frac{t^m(a)_m}{m!(b)_m}\,{}_2F_1\left[\begin{matrix}-m,1-b-m;\dfrac{x-1}{x}\\ 1-a-m\end{matrix}\right].$$

In (2.2), putting ${}_1F_1(a;b;t)=M(a;b;t)$ and using ${}_0F_0(-;-;-t)=e^{-t}$, the generating function (2.1) is obtained.

3. Semi-Orthogonal Relations

The semo-orthogonal relations to be extablished are

$$(3.1)\qquad \int_1^{\infty}x^{b+2n-2}(1-x)^{-a-n}\,{}_2F_1\left(\begin{matrix}-m,1-b-m;\dfrac{x-1}{x}\\ 1-a-m\end{matrix}\right)\times{}_2F_1\left(\begin{matrix}-n,1-b-n;\dfrac{x-1}{x}\\ 1-a-n\end{matrix}\right)dx$$

$$(3.1a)\qquad =0,\quad \text{if } m<n$$

$$(3.1b)\qquad =(-1)^n\frac{n!(1-b-n)_n\Gamma(1-a-n)\Gamma(b-1)}{(1-a-n)_n\Gamma(b-a)},\quad \text{if } m=n$$

where $Re a < 1 - n$, $Re b < n - m + 1$, $Re b > 1 + m - n$.

$$\int_1^\infty x^{b+2n-2}(1-x)^{-a-n}\,{}_2F_1\left(\begin{matrix}-m, 1-b-m; \dfrac{x-1}{x}\\ 1-a-m\end{matrix}\right)$$

$$\times\,{}_2F_1\left(\begin{matrix}-n, 1-b-n; \dfrac{x-1}{x}\\ 1-a-n\end{matrix}\right) dx \tag{3.2}$$

$$= 0, \quad \text{if } m < n \tag{3.2a}$$

$$= \frac{n!(1-a-n)_n\Gamma(1-a-n)\Gamma(1+a-b)}{(1-a-n)_n\Gamma(2-b)}, \quad \text{if } m = n \tag{3.2b}$$

where $Re(a-b) > -1$, $Re a < 1 - n$, $Re b < 1$.

$$\int_0^\infty x^{b+2n-2}(x+1)^{-a-n}\,{}_2F_1\left(\begin{matrix}-m, 1-b-m; \dfrac{x+1}{x}\\ 1-a-m\end{matrix}\right)$$

$$\times\,{}_2F_1\left(\begin{matrix}-n, 1-b-n; \dfrac{x+1}{x}\\ 1-a-n\end{matrix}\right) dx \tag{3.3}$$

$$= 0, \quad \text{if } m < n \tag{3.3a}$$

$$= (-1)^n\frac{n!(1-b-n)_n\Gamma(1+a-b)\Gamma(b-1)}{(1-a-n)_n\Gamma(a+n)}, \quad \text{if } m = n \tag{3.3b}$$

where $Re(a-b) > -1$, $Re b < n - m + 1$, $Re b > 1 + m - n$.

Proof. To prove (3.1), we write its left hand side in the form:

$$\sum_{r=0}^{m}\frac{(-m)_r(1-b-m)_r}{r!(1-a-m)_r}(-1)^r \tag{3.4}$$

$$\times\int_0^1 x^{b+2n-r-2}\,{}_2F_1\left(\begin{matrix}-n, 1-b-n; \dfrac{x-1}{x}\\ 1-a-n\end{matrix}\right) dx$$

On evaluating the integral in (3.4) with the help of (1.2), it reduces of the form

(3.5)

$$\sum_{r=0}^{m}\frac{(-m)_r(1-b-m)_r}{r!(1-a-m)_r}(-1)^r\frac{\Gamma(n+b-r-1)\Gamma(1-a+r-n)(-r)_n}{(1-a-n)_n\Gamma(b-a)}$$

If $r < n$, the numerator of (3.5) vanishes, and since r runs from 0 to m, it follows that (3.5) also vanish, which proves (3.1a).

When $m = n$, using the standard result [1, p.274].

$$(3.6) \qquad (-r)_n = \begin{cases} \frac{(-1)^n r!}{(r-n)!}, & \text{if } 0 \leq n \leq r \\ 0 \quad , & \text{if } n > r \end{cases}$$

and simplifying, the right hand side of (3.1b) follows from (3.5).

To prove (3.2), first we reduce its left hand side to the form similar to (3.4) , then evaluate the last integral with the help of the integral (1.7) to obtain

$$(3.7) \qquad \sum_{r=0}^{m} \frac{(-m)_r(1-b-m)_r}{r!(1-a-m)_r} \frac{\Gamma(1+a-b)\Gamma(1-a-n+r)(-r)_n}{\Gamma(2+r-b-n)(1-a-n)_n}.$$

We see that (3.7) is of the same form as (3.5). Therefore, for $m < n$ all terms of (3.7) vanish, which proves (3.2a).

When $m = n$, using the standard result (3.6) and simplifying, the right hand side of (3.2b) follows from (3.7).

To prove (3.3), we first reduce its left hand side to the form similar to (3.4), then evaluate the last integral with the help of the integral (1.9) to get

(3.8)
$$\sum_{r=0}^{m} (-1)^r \frac{(-m)_r(1-b-m)_r}{r!(1-a-m)_r} (-1)^n \frac{\Gamma(b-1+n-r)\Gamma(1+a-b)(-r)_n}{(1-a-n)_n\Gamma(a+n-r)}.$$

Since (3.8) is of the same form as (3.5). Therefore, for $m < n$ all terms of (3.8) vanish, which proves (3.3a).

When $m = n$, using the standard result (3.6) and simplifying, the right hand side of (3.3b) follows from (3.8).

Note: On continuing as above, we can find the values of the integrals (3.1), (3.2) and (3.3) for $m = n+1, n+2, n+3, \cdots$.

4. Finite series expansions involving the hypergeometric functions

Based on the relations (3.1a) and (3.1b), (3.2a) and (3.2b), and (3.3a) and (3.3b), we can generate a theory concering the expansion of arbitrary polynomials, or functions in general, in a finite series expansion of the hypergeometric polynomials. Specially if $F(x)$, $G(x)$ and $H(x)$ are suitable functions defined for all x, we consider for expansions of the general form

(4.1)

$$F(x) = \sum_{m=0}^{n} A_m \, {}_2F_1 \left(\begin{matrix} -m, 1-b-m; \dfrac{x-1}{x} \\ 1-a-m \end{matrix} \right), \; 0 < x < 1, \, m \leq n$$

where the expansion coefficients A_m are given by

(4.2)

$$A_m = (-1)^m \frac{(1-a-m)_m \Gamma(b-a)}{m!(1-b-m)_m \Gamma(1-a-m)\Gamma(b-1)} \times \int_0^1 F(x) x^{b+2m-2} (1-x)_2^{-a-m} F_1 \left(\begin{matrix} -m, 1-b-m; \dfrac{x-1}{x} \\ 1-a-m \end{matrix} \right) dx$$

(4.3)

$$G(x) = \sum_{m=0}^{n} B_m \, {}_2F_1 \left(\begin{matrix} -m, 1-b-m; \dfrac{x-1}{x} \\ 1-a-m \end{matrix} \right), \; 1 < x < \infty, \, m \leq n$$

where the expansion coefficients B_m are given by

(4.4)

$$B_m = \frac{(1-a-m)_m \Gamma(2-b)}{m!(1-b-m)_m \Gamma(1-a-m)\Gamma(1+a-b)} \times \int_1^\infty G(x) x^{b+2m-2} (x-1)_2^{-a-m} F_1 \left(\begin{matrix} -m, 1-b-m; \dfrac{x-1}{x} \\ 1-a-m \end{matrix} \right) dx$$

(4.5)

$$H(x) = \sum_{m=0}^{n} C_m \, {}_2F_1 \left(\begin{matrix} -m, 1-b-m; \dfrac{x+1}{x} \\ 1-a-m \end{matrix} \right), \; 0 < x < \infty, \, m \leq n$$

where the expansion coefficients C_m are given by

(4.6)

$$C_m = (-1)^m \frac{(1-a-m)_m \Gamma(a+m)}{m!(1-b-m)_m \Gamma(1+a-b)\Gamma(b-1)} \times \int_0^\infty H(x) x^{b+2m-2} (x+1)_2^{-a-m} F_1 \left(\begin{matrix} -m, 1-b-m; \dfrac{x+1}{x} \\ 1-a-m \end{matrix} \right) dx$$

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