Honam Mathematical J. 16(1994), No. 1,pp. 9-15

THE VANISHING OF THE GENERALIZED WHITEHEAD PRODUCTS AND T-SPACES

HONG-CHAN SON

Dept. of Mathematics, Chonbuk National University, Chonju 560-756, Korea.

1. Introduction

The original Whitehead product is a pairing $\pi_p\ (X) \times \pi_q\ (X) \longrightarrow \pi_{p+q-1}\ (X)$ and the generalized Whitehead prodcut (written GWP) is a pairing $\pi\ (\Sigma A, X) \times \pi\ (\Sigma B, X) \longrightarrow \pi\ (\Sigma(A \wedge B), X)$ ([3],[5],[8]). In case A and B are spheres, the GWP is essentially the Whitehead product.

In [3], the author established some properties of the GWP and investigated spaces in which all GWPs vanish, and he remarked that such spaces are not necessarily H-spaces. J.Aguadé ([2]) introduced and studied a T-space, which is a space X having the property that the space of free loops on X is equivalent in some sense to the product of X by the space of based loops on X. It is known that ([2],[7]) an H-space is a T-space but the converse is not true.

In this paper, we show that all the GWPs vanish in T-space by using the notion of cyclic maps. And as a corollary to this fact we obtain that every T-space is a W-space.

Throughout this paper, space means a space of homotopy type of connected locally finite CW-complex. We also assume that spaces have non-degenerate base point. All maps shall mean continuous functions and all homotopies and maps are to respect base points(except for the case of the mapping space X^A). The base point and constant map at base point will be denoted by $*$ and 0 respectively. For simplicity, sometimes we use the same symbol for a map and its homotopy class. We denote $\pi(X,Y)$ the set of homotopy classes of pointed maps $X \longrightarrow Y$. Let ΣA denote the reduced suspension of A and ΩB the loop space of B. And let $A \wedge B$ be the smash product.

Received April 27, 1994. Revised Jun 24, 1994.
Research supported by TGRC-Kosef (TGRC Post Doc. Program) in 1993.

2. The generalized Whitehead product

A map $f : A \longrightarrow X$ is called cyclic, if there exists a map $m : A \times X \longrightarrow X$ such that $mj \simeq \nabla(f \vee 1)$, where $j : A \vee X \longrightarrow A \times X$ is the inclusion and $\nabla : X \vee X \longrightarrow X$ is the folding map defined by $\nabla(x, *) = x = \nabla(*, x)$. The Gottlieb set G(A,X) is the set of all homotopy classes of cyclic maps from A to X.

PROPOSITION 2.1 ([10],[11]).

(1) *If A is a co $-$ H-group, then $G(A, X)$ is contained in the center of $\pi(A, X)$*

(2) *X is an H- space if and only if $G(A, X) = \pi(A, X)$ for any space A*

Let $\alpha = [f] \in \pi(\Sigma A, X)$ and $\beta = [g] \in \pi(\Sigma B, X)$ and let $f' = f \circ \Sigma p_A : \Sigma(A \times B) \longrightarrow X$ and $g' = g \circ \Sigma p_B : \Sigma(A \times B) \longrightarrow X$ where $p_A : A \times B \longrightarrow A$ and $p_B : A \times B \longrightarrow B$ are projections. We define the commutator

$$k = (f'^{-1} \cdot g'^{-1}) \cdot (f' \cdot g') : \Sigma(A \times B) \longrightarrow X$$

where the products and inverses come from the suspension structure of $\Sigma(A \times B)$. Since $k|\Sigma(A \times *) \simeq 0$ and $k|\Sigma(* \times B) \simeq 0$, we have $k|\Sigma(A \vee B) \simeq 0$II. Now consider a cofibrationII

$$\Sigma(A \vee B) \xrightarrow{\Sigma j} \Sigma(A \times B) \xrightarrow{\Sigma q} \Sigma(A \wedge B)$$

where $j : A \vee B \longrightarrow A \times B$ is the inclusion and $q : A \times B \longrightarrow A \wedge B$ is the quotient map. Then we have the Puppe exact sequence

$$\pi(\Sigma(A \wedge B), X) \xrightarrow{(\Sigma q)^*} \pi(\Sigma(A \times B), X) \xrightarrow{(\Sigma j)^*} \pi(\Sigma(A \vee B), X)$$

$k|\Sigma(A \vee B) \simeq 0$ means that $(\Sigma j)^*(k) = 0$, thus there exists $\tilde{k} \in \pi(\Sigma(A \wedge B), X)$ such that $(\Sigma q)^*(\tilde{k}) = k$, i.e., $\tilde{k}\Sigma q = k$.

And we know that the homotopy class of $\tilde{k}$ does not depend on the choice of the map k([3]).

DEFINITION 2.2. The GWP of $\alpha = [f] \in \pi(\Sigma A, X)$ and $\beta = [g] \in \pi(\Sigma B, X)$ is defined to be $[\alpha, \beta] = [\tilde{k}] \in \pi(\Sigma(A \wedge B), X)$.

Consider two maps $f : (CA, A) \longrightarrow (X, *)$ and $g : (CB, B) \longrightarrow (X, *)$ where $CA = A \times I / A \times 1 \cup * \times I$ is the reduced cone on A. And consider the subspace $Q = CA \times B \cup A \times CB$ of $CA \times CB$ and define $h : Q \longrightarrow X$ by

$$h((a,t),b) = f(a,t)$$
$$h(a,(b,u)) = g(b,u)$$

where $a \in A$, $b \in B$ and $t, u \in I$.

Now there is a homeomorphism $\nu : A * B \longrightarrow Q$ defined by

$$\nu(a,b,t) = \begin{cases} (a,(b,1-2t)), & \text{for } 0 \leq t \leq \frac{1}{2} \\ ((a,2t-1),b), & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

where $A * B$ is the join of A and B.

Let P be the subspace of $A * B$ consisting of all points $(a, *, t)$ and all points $(*, b, u)$. Then P is contractible and $A * B/P$ is $\Sigma(A \wedge B)$ and so the projection $\mu' : A * B \longrightarrow \Sigma(A \wedge B)$ is a homotopy equivalence. Let $\mu : \Sigma(A \wedge B) \longrightarrow A * B$ denote the homotopy inverse of μ'. Thus the map gives rise to a map $h\nu\mu : \Sigma(A \wedge B) \longrightarrow X$ and it is easily seen that the homotopy class of $h\nu\mu$ does not depend on the representatives f and g of α and β.

PROPOSITION 2.3 ([3]). *The GWP $[\alpha, \beta]$ of $\alpha \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$ is equal to $[h\nu\mu]$.*

Suppose that X is an H-space, then the group $\pi(\Sigma(A \times B), X)$ is abelian and so $k = f'^{-1} \cdot g'^{-1} \cdot f' \cdot g' \simeq 0$. Therefore, all GWPs vanish in X if X is an H-space.

Let $j_0 : \Sigma A \vee \Sigma B \longrightarrow \Sigma A \times \Sigma B$ be the inclusion map and $\tilde{k}_0 : \Sigma(A \wedge B) \longrightarrow \Sigma A \vee \Sigma B$ the map of Definition 2.2 which represents the GWP of the class of the inclusion maps $i_1 : \Sigma A \longrightarrow \Sigma A \vee \Sigma B$ and $i_2 : \Sigma B \longrightarrow \Sigma A \vee \Sigma B$. $\tilde{k}_0$ is called the GWP map.

PROPOSITION 2.4 ([3]). *If $p : \Sigma A \times \Sigma B \longrightarrow \Sigma A \wedge \Sigma B$ is the projection, then for any space X, the following sequence of pointed set is exact*

$$\pi(\Sigma A \wedge \Sigma B, X) \xrightarrow{p^*} \pi(\Sigma A \times \Sigma B, X) \xrightarrow{j_0{}^*} \pi(\Sigma A \vee \Sigma B, X) \xrightarrow{\tilde{k}_0{}^*} \pi(\Sigma(A \wedge B), X)$$

THEOREM 2.5. *For $\alpha = [f] \in \pi(\Sigma A, X)$ and $\beta = [g] \in \pi(\Sigma B, X)$, $[\alpha, \beta] = 0$ if and only if there is a map $m : \Sigma A \times \Sigma B \longrightarrow X$ such that $mj_0 \simeq \nabla(f \vee g)$.*

Proof. Define $l : \Sigma A \vee \Sigma B \longrightarrow X$ by $l((a,t),*) = f(a,t)$ and $l(*,(b,u)) = g(b,u)$ for $a \in A, b \in B$ and $t, u \in I$.

Let $h_0 : Q \longrightarrow \Sigma A \vee \Sigma B$ be the map given by $h_0((a,t),b) = i_1(a,t)$ and $h_0(a,(b,u)) = i_2(b,u)$, and let h, ν, μ be the maps of Proposition 2.3, where $Q = CA \times B \cup A \times CB$. Then $lh_0 = h$ and $l\tilde{k}_0 = lh_0\nu\mu = h\nu\mu$. This means that $[l\tilde{k}_0] = [\alpha, \beta]$. By Proposition 2.4 $l\tilde{k}_0 \simeq 0$ if and only if there exists $m : \Sigma A \times \Sigma B \longrightarrow X$ with ${j_0}^*m = l$. But $l = \nabla(f \vee g)$. This gives $mj_0 \simeq \nabla(f \vee g)$.

PROPOSITION 2.6. *If $\alpha, \beta \in \pi(\Sigma A, X)$ and $d : A \longrightarrow A \wedge A$ is the composition of diagonal map $\Delta : A \longrightarrow A \times A$ and projection $q : A \times A \longrightarrow A \wedge A$. Then $(\Sigma d)^*[\alpha, \beta] = (\alpha, \beta)$, the commutator of α and β.*

Proof. For $\alpha = [f], \beta = [g] \in \pi(\Sigma A, X)$, let $k = {f'}^{-1} \cdot {g'}^{-1} \cdot f' \cdot g' : \Sigma(A \times A) \longrightarrow X$ and $\tilde{k}$ be such that $k = \tilde{k}\Sigma q$ as in the definition of the GWP. Note that $(\Sigma d)^*[\alpha, \beta] = [\tilde{k}\Sigma d]$ and $\Sigma d = \Sigma q \Sigma \Delta$. So $\tilde{k}\Sigma d = \tilde{k}\Sigma q \Sigma \Delta = k\Sigma\Delta$. But it can be checked that $k\Sigma\Delta(a,t) = (f^{-1} \cdot g^{-1} \cdot f \cdot g)(a,t)$. Thus we obtain that $(\Sigma d)^*[\alpha, \beta] = (\alpha, \beta)$.

3. The vanishing of the GWPs and T-spaces

In this section, we investigate the situation that the GWPs vanish.

For the natural isomorphism $\tau : \pi(\Sigma X, Y) \longrightarrow \pi(X, \Omega Y)$, we denote e by

$\tau^{-1}(1_{\Omega X}) : \Sigma\Omega X \longrightarrow X$.

PROPOSITION 3.1 ([3]). *All GWPs vanish in X if and only if $\pi(\Sigma P, X)$ is abelian for all space P.*

Proof. For any space P, and any $\alpha, \beta \in \pi(\Sigma P, X)$, $[\alpha, \beta] = 0$ implies $(\alpha, \beta) = 0$ by Proposition 2.6. Thus $\pi(\Sigma P, X)$ is abelian.

Conversely, if $\pi(\Sigma P, X)$ is abelian for all P, then, in particular, $\pi(\Sigma(A \times B), X)$ is abelian. So the commutator

$$k = {f'}^{-1} \cdot {g'}^{-1} \cdot f' \cdot g' : \Sigma(A \times B) \longrightarrow X$$

is nullhomotopic. And so that $[\tilde{k}] = [\alpha, \beta] = 0$.

The notion *nil* $\Omega X \leq 1$ is the assertion that the commutator map $\Omega X \times \Omega X \longrightarrow \Omega X$ is nullhomotopic. Then we can see *nil* $\Omega X \leq 1$ implies $\pi(P, \Omega X)(\approx \pi(\Sigma P, X))$ is abelian for all P and so all GWPs vanish in X by the preceding proposition. Berstein and Ganea ([4]) showed that there is a space with *nil* $\Omega X \leq 1$ which is not an H-space.

Let us consider the fibration

$$\Omega X \longrightarrow X^{S^1} \xrightarrow{p} X$$

where X^{S^1} is the free loop space of X and p is the evaluation at 1.

DEFINITION 3.2. A space X is a T-space if the fibration

$$\Omega X \longrightarrow X^{S^1} \longrightarrow X$$

is fibre homotopy equivalent to the trivial fibration

$$\Omega X \longrightarrow X \times \Omega X \longrightarrow X.$$

PROPOSITION 3.3 ([2]). *If X is a T-space if and only if the map $\nabla(e \vee 1) : \Sigma\Omega X \vee X \longrightarrow X$ can be extended up to homotopy to a map $F : \Sigma\Omega X \times X \longrightarrow X$.*

Note that the fact $\nabla(e \vee 1) : \Sigma\Omega X \vee X \longrightarrow X$ has a homotopy extension to $F : \Sigma\Omega X \times X \longrightarrow X$ means that $e : \Sigma\Omega X \longrightarrow X$ is cyclic.

THEOREM 3.4. *X is a T-space if and only if $G(\Sigma A, X) = \pi(\Sigma A, X)$ for any space A.*

Proof. Let $f : \Sigma A \longrightarrow X$ be any map. If X is a T-space, then by the preceding theorem, there is a map $F : \Sigma\Omega X \times X \longrightarrow X$ such that $\nabla(e \vee 1) \simeq Fj$. Note that $f = e \circ \Sigma\tau(f)$. Define $H : \Sigma A \times X \longrightarrow X$ by $H = F \circ (\Sigma\tau(f) \times 1)$. It can be checked that $Hj \simeq \nabla(f \vee 1)$, where $j : \Sigma A \vee X \longrightarrow \Sigma A \times X$ is the inclusion. This means that $f : \Sigma A \longrightarrow X$ is cyclic, so that $[f] \in G(\Sigma A, X)$.

Conversely, suppose $G(\Sigma A, X) = \pi(\Sigma A, X)$ for all space A. If we take $A = \Omega X$, then $G(\Sigma\Omega X, X) = \pi(\Sigma\Omega X, X)$. Thus we have $e : \Sigma\Omega X \longrightarrow X$ is cyclic, so X is a T-space.

For all space A, ΣA is a $co-H$-group. Thus from Proposition 2.1 (1), Proposition 3.1 and Theorem 3.4, we obtain the following

THEOREM 3.5. *If X is a T-space then all GWPs vanish in X.*

It is proven ([1]) that only S^1,S^3 and S^7 are T-spaces. In fact H-spaces, T-spaces and G-spaces are equivalent in the category of spheres ([12]). Thus we have the following

Corollary 3.6. *All GWPs vanish in S^1,S^3 and S^7.*

In case A is a sphere S^n, the Gottlieb set $G(A,X)$ reduced to the Gottlieb group([6]) $G_n(X) = G(S^n, X)$. And $G_n(X)$ is an abelian subgroup of the homotopy group $\pi_n(X)$ by Proposition 2.1 (1). Let $P_n(X)$([6]) be the set of elements $[f]$ in $\pi_n(X)$ whose Whitehead product with all elements of all homotopy groups is zero. It turns out ([6] or Theorem 2.5 of [11]) that $P_n(X)$ form a subgroup of $\pi_n(X)$.

Definition 3.7.

(1) X is a G-space if $G_n(X) = \pi_n(X)$ for all n
(2) X is a W-space if $P_n(X) = \pi_n(X)$ for all n

Theorem 3.8.

(1) *An H-space is a T-space*
(2) *A T-space is a G-space*
(3) *A G-space is a W-space*

Proof.
(1) By Proposition 2.1,(2) and Theorem 3.4. (2) Note that ΣS^n is homeomorphic to S^{n+1} and use Theorem 3.4. (3) It suffices to show that $G_n(X) \subset P_n(X)$. Suppose that $f : S^n \longrightarrow X$ is cyclic. Then there exists $m : S^n \times X \longrightarrow X$ such that $mj \simeq \nabla(f \vee 1)$, where $j : S^n \vee X \longrightarrow S^n \times X$ is the inclusion and $\nabla : X \vee X \longrightarrow X$ is the folding map. Let $g : S^m \longrightarrow X$ be any map. Then it can be shown that $m(1 \times g) : S^n \times S^m \longrightarrow X$ satisfies $m(1 \times g)j \simeq \nabla(f \vee g)$. By Theorem 2.5, the Whitehead product of the classes of f and g is zero. That is, the class of f is in $P_n(X)$.

It is known([2],[7]) that there exist T-spaces which are not H-spaces. And it is also known([6],[7]) that the converses of (2) and (3) are not true either.

References

1. J.Aguadé, *On the space of free loops of an odd sphere*, Publ.Mat.UAB **25** (1981), 87–90.
2. J.Aguadé, *Decomposable free loops spaces*, Canad. J. Math. **39** (1987), 938–955.
3. M. Arkowitz, *The generalized Whitehead product*, Pacific J.Math. **12** (1962), 7–23.
4. I.Berstein and T.Ganea, *Homotopical nilpotency*, Ill.Jour. Math. **5** (1961), 99–130.
5. D.E.Cohen, *Products and carrier theory*, Proc. Lond. Soc.(3) **7** (1957), 219–248.
6. D.H.Gottlieb, *Evaluation subgroups of homotopy groups*, Amer.J. Math. **91** (1969), 729–756.

7. H.B.Haslam, *G-spaces and H-spaces*, Ph.D. dissertation, University of California, Irvine, 1969.
8. P.J.Hilton, *Homotopy Theory and Duality*, Gordon and Breach Science Publishers, Inc., 1965.
9. K.L.Lim, *On cyclic maps*, J.Aust.Math. Soc.(Series A) **32** (1982), 349–357.
10. ———, *On evaluation subgroups of generalized homotopy groups*, Canad. Math.Bull. **27** (1984), 78–86.
11. N.Oda, *The homotopy set of the axes of pairings*, Canad. J. Math. **52** (1990), 856-868.
12. M.H.Woo and Y.S.Yoon, *T-spaces by Gottlieb groups and its duality*, to appear in J.Austral. Math.Soc.(Series A).