Honam Mathematical J. 16(1994), No. 1,pp. 1–8

SOME REMARKS ON HOMOTOPY GROUPS OF Spin(n) AND ^{C}SPINORIAL VECTOR BUNDLES

KEE-AN LEE, HONG-JAE LEE AND WON-KEE JEON

Dept. of Mathematics, Jeonju Woo-Suk University, Chonbuk 565-800, Korea.
Dept. of Mathematics, Chonbuk National University, Chonbuk 560-765, Korea.

In the study of the index theorem we sometimes need homotopy groups of Spin(n) and some properties of ^{C}spinorial vector bundles. These are proved in the theorem 3 and in the theorem 7 of this note, respectively.

We shall use the notations $\mathbf{Z}$ (integers), $\mathbf{R}$ (reals), $\mathbf{C}$ (complexes), and for $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in \mathbf{R}^n$,

$$(x \,|\, y) = x_1 y_1 + \cdots + x_n y_n.$$

Let $C(\mathbf{R}^n, -(x\,|\,y)) = C_n$ be the Clifford algebra of the quadratic form $(\mathbf{R}^n, -(x\,|\,y))$.

DEFINITION 1. Pin(n) is the subgroup of the multiplicative group of units in C_n generated by the $(n-1)$-dimensional sphere S^{n-1} ($S^{n-1} \subset \mathbf{R}^n \subset C_n$). Spin($n$) is the subgroup $\mathrm{Pin}(n) \cap C_n^0$ of Pin(n), where $C_n = C_n^0 \oplus C_n^1$.

PROPOSITION 2. *The map $\phi\colon Pin(n) \longrightarrow O(n)$ is a continuous group epimorphism with $\phi^{-1}(SO(n)) = Spin(n)$ and $\ker\phi = \{+1, -1\}$, where for all $x \in \mathbf{R}^n$ and $u \in Pin(n)$*

$$\phi(u)(x) = ux\,{}^t u$$

and for $u = u_1 \cdots u_r$ ($u_i \in S^{n-1}$ for $i = 1, \cdots r$) ${}^t u = u_r \cdots u_1$.

Proof. Note that for each $u = u_1 \cdots u_r$ ($u_i \in S^{n-1}$) and $x \in \mathbf{R}^n$

$$\|\phi(u)x\|^2 = (ux\,{}^t u)(ux\,{}^t u) = ux \cdot x\,{}^t u = x \cdot x = \|x\|^2$$

Received March 2, 1993.

The present stueies were supported by the Basic Science Research Institute Program, Ministry of Education, 1993, Project # 109.

which implies that $\phi(u) \in O(n)$. For $u, v \in \mathrm{Pin}(n)$ and $x \in \mathbb{R}^n$

$$\begin{aligned}\phi(uv)x &= uvx{}^tv{}^tu = u(\phi(v)x){}^tu \\ &= \phi(u)\phi(v)(x).\end{aligned}$$

that is, $\phi(uv) = \phi(u)\phi(v)$ and thus ϕ is a group homomorphism.

To prove that ϕ is an epimorphism we prove that $\phi(u)$ for $u \in S^{n-1}$ is a reflection through the hyperplane perpendicular to u. For all $x \in \mathbb{R}^n$ we have the decomposition

$$x = tu + u' \qquad (u \perp u').$$

Therefore we have the following :

$$\phi(u)x = u(tu + u'){}^tu) = tuu{}^tu + uu'{}^tu = -tu + u'.$$

Thus $\phi(u)$ is a reflection through the hyperplane perpendicular to u if $u \in S^{n-1}$. Since these reflections generate $O(n)$, and thus ϕ is surjective.

For $u \in \ker\phi$ we have $ue_i^tu = e_i$ or $ue_i = e_iu$ since $u^tu = 1$, where $e_1, \cdots, e_n$ are orthonormal basis of $\mathbb{R}^n$.

Conversely, these conditions imply that $u \in \ker\phi$. That is, $u \in \ker\phi$ implies that u is in the center of C_n for n even which is $\mathbb{R}1$ ([2]).

Thus we have the following :

$$u \in \ker\phi \Longleftrightarrow u \in \mathbb{R}1 \quad \text{and} \quad u \cdot u = 1.$$

Therefore $\ker\phi = \{+1, -1\}$. Moreover, for $u \in S^{n-1}$, $\det(\phi(u)) = -1$ and $\det(\phi(u_1 \cdots u_r)) = (-1)^r$ for $u_i \in S^{n-1}$ $(i = 1, \cdots, r)$. Thus

$$u \in \mathrm{Spin}(n) \Longleftrightarrow \phi(u) \in SO(n).$$

In consequence, ϕ is locally trivial and thus ϕ is a continuous group homomorphism.

We have two exact sequences of topological groups as follows ([1], [2], [3], [6]).

$$(*) \qquad \begin{array}{l} 1 \longrightarrow \{+1, -1\} \longrightarrow \mathrm{Pin}(n) \stackrel{\phi}{\longrightarrow} O(n) \longrightarrow 1 \\ 1 \longrightarrow \{+1, -1\} \longrightarrow \mathrm{Spin}(n) \stackrel{\phi}{\longrightarrow} SO(n) \longrightarrow 1 \end{array}$$

THEOREM 3. *For* $i \geq 1$, $\pi_i(Spin(1)) = 0$, $\pi_2(Spin(2)) = \mathbf{Z}_2$ *and for* $i \geq 1$, $\pi_i(Spin(2)) = 0$. *For* $n \geq 3$,

$$\pi_i(Spin(n)) = \begin{cases} \mathbf{Z}_2 & \text{if } i = 1 \\ 0 & \text{if } i = 2. \end{cases}$$

Moreover, $\pi_3(Spin(n)) = \mathbf{Z}$ *for* $n \geq 5$ *and* $\pi_0(Spin(n)) = 0$ *for* $n \geq 3$.

Proof. We have the following table about homotopy groups ([2],[6]):

$$\begin{array}{ll} \pi_i(O(1)) = 0 = \pi_i(SO(1)) & \text{for } i \geq 1 \\ \pi_1(O(2)) = \mathbf{Z} = \pi_1(SO(2)) & \\ \pi_i(O(2)) = 0 = \pi_i(SO(2)) & \text{for } i \geq 3 \\ \pi_1(O(n)) = \mathbf{Z}_2 = \pi_1(SO(n)) & \text{for } n \geq 3 \\ \pi_2(O(n)) = 0 = \pi_2(SO(n)) & \text{for } n \geq 3 \\ \pi_3(O(n)) = \mathbf{Z} = \pi_3(SO(n)) & \text{for } n \geq 5. \end{array}$$

Moreover, by the exact sequence of the homotopy groups of $(*)$, we have

$$\pi_i(SO(n)) = \pi_i(\mathrm{Spin}(n)) \qquad \text{for } i \geq 1.$$

Thus we have the results.

To prove the last statements it suffices to prove that $+1$ and -1 in Spin(n) ($n \geq 3$) are connected since $\phi\colon \mathrm{Spin}(n) \longrightarrow SO(n)$ is locally trivial. Note that

$$(e_1 \cos t + e_2 \sin t)(e_1 \cos t - e_2 \sin t) = \cos 2t - e_1 e_2 \sin 2t$$

is a member of Spin(n) where $\{e_1, \cdots, e_n\}$ is an orthonormal basis of $\mathbf{R}^n$. Then we get a path in Spin(n) ($n \geq 3$) from $+1$ to -1 for $0 \leq t \leq \frac{\pi}{2}$. Therefore, Spin($n$) is a path connected space, and thus $\pi_0(\mathrm{Spin}(n)) = 0$ for $n \geq 3$.

For a topological group G and a topological space X a G-cocycle on X is given by an open cover $\{U_j\}$ of X, and continuous maps

$$g_{ji}\colon U_i \cap U_j \longrightarrow G \qquad \cdot \ni \cdot \quad g_{kj}(x) g_{ji}(x) = g_{ki}(x)$$

for all $x \in U_i \cap U_j \cap U_k$.

Two G-cocycles $\{U_j, g_{ji}\}$ and $\{V_s, h_{sr}\}$ are *equivalent* if there exist continuous maps

$$g_i^r : U_i \cap V_r \longrightarrow G \quad \cdot \ni \cdot \quad g_j^s(x) g_{ji}(x) g_i^r(x)^{-1} = h_{sr}(x)$$

for each $x \in U_i \cap U_j \cap V_r \cap V_s$. The quotient set of the set of all G-cocycles by the above equivalence relation will be denoted by $H^1(X; G)$.

Let $\Phi_n^k(X)(k = \mathbb{R}$ or $\mathbb{C})$ be the set of isomorphism classes of k-vector bundles of rank n over a compact space X. Then $\Phi_n^k(X)$ is naturally isomorphic to the set $H^1(X : G)$, where $G = GL_n(k)$ ([2]).

DEFINITION 4. Let V be real vector bundle over a compact space X. *An orientation* on V is an element $\alpha \in H^1(X : SL_n(\mathbb{R}))$ whose image under the map : $H^1(X : SL_n(\mathbb{R})) \longrightarrow H^1(X : GL_n(\mathbb{R}))$ induced by the inclusion $SL_n(\mathbb{R}) \hookrightarrow GL_n(\mathbb{R})$ is the class of the bundle V. *A spinorial structure on* V is an element $\beta \in H^1(X : \mathrm{Spin}(n))$ whose image under the composition

$$H^1(X : \mathrm{Spin}(n)) \longrightarrow H^1(X : SO(n)) \longrightarrow H^1(X : GL_n(\mathbb{R}))$$

is a class of V.

Let V be a real vector bundle of rank n over a compact space X. Then V may be provided with an orientation (resp., a spinorial structure) if $V \cong P \times_G \mathbb{R}^n$ where $G = SO(n)$ (resp. $\mathrm{Spin}(n)$) and P is a principal bundle with structural group $SO(n)$ (resp. $\mathrm{Spin}(n)$) ([2]). In fact, this bundle P is associated with $\alpha \in H^1(X : SO(n))$ (resp. $\beta \in H^1(X : \mathrm{Spin}(n))$) defined in definition 4 ([2]).

EXAMPLE 5. (i) Let W be a real vector bundle, and let P be a principal bundle with $O(n)$ such that

$$W = P \times_{O(n)} \mathbb{R}^n.$$

In this case, the principal bundle P is associated with the $O(n)$-cocycle $\{g_{ji}\}$, i. e., $g_{ji}(x) \in O(n)$ for each $x \in U_j \cap U_i$ and $\{U_i\}$ is an open cover of X (X is compact). We put

$$h_{ji} = \begin{pmatrix} g_{ji}(x) & 0 \\ 0 & g_{ji}(x) \end{pmatrix}.$$

Then $\{h_{ji}\}$ is a $SO(n)$-cocycle and thus $W \oplus W$ may be written as $P' \times_{O(n)} \mathbb{R}^n$, where P' is a principal bundle associated with $SO(2n)$-cocycle $\{h_{ji}\}$. Hence $W \oplus W$ is an oriented bundle ([3]).

(ii) Let W be an oriented vector bundle. Then there exists a principal bundle P associated with $SO(n)$-cocyle $\{ g_{ji} \}$ such that

$$W = P \times_{SO(n)} \mathbb{R}^n.$$

Moreover, $V = W \oplus W$ is a vector bundle with spinorial structure (V is said ***a spinorial vector bundle***). By (i), V is an oriented vector bundle, and thus

$$V = P' \times_{SO(n)} \mathbb{R}^{2n},$$

where P' is a principal bundle associated with $SO(n)$-cocycle $\{\phi(D(\bar{g}_{ji}, \bar{g}_{ji})\}$, where ϕ, D and $\bar{g}_{ji}$ are defined as follows.

$\phi \colon \mathrm{Spin}(2n) \longrightarrow SO(2n)$ is defined in proposition 2.

$D \colon \mathrm{Spin}(n) \times \mathrm{Spin}(n) \longrightarrow \mathrm{Spin}(2n)$ is the homomorphism definde by

$$D(\alpha_1 \cdot \alpha_2) = i_1(\alpha_1) i_2(\alpha_2) \qquad \forall \alpha_1, \alpha_2 \in \mathrm{Spin}(n)$$

where i_1 (resp. i_2) is induced by the canonical inclusion of $\mathbb{R}^n \oplus 0$ (resp. $0 \oplus \mathbb{R}^n$) in $\mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{R}^{2n}$. For $u \in SO(n)$ and $\bar{u} \in \mathrm{Spin}(n)$ with $\phi(\bar{u}) = u$, $D(\bar{u}, \bar{u})$ is well-defined element of Spin($2n$). Then it is clear that $\{ \phi(D(\bar{g}_{ji}, \bar{g}_{ji}) \}$ is a $SO(2n)$-cocycle. Therefore

$$V = P' \times_{\mathrm{Spin}(n)} \mathbb{R}^{2n}$$

is a spinorial vector bundle.

PROPOSITION 6. *Let V and V' be spinorial vector bundles over a compact space X ($\dim_{\mathbb{R}}(V) = n$, $\dim_{\mathbb{R}}(V') = n'$). Then $V \oplus V'$ is a spinorial vector bundle of rank $n + n'$.*

Proof. Note that we have the commutative diagram

$$\begin{array}{ccc}
\mathrm{Spin}(n) \times \mathrm{Spin}(n') & \longrightarrow & \mathrm{Spin}(n+n') \\
\downarrow & & \downarrow \\
SO(n) \times SO(n') & \copyright & SO(n+n') \\
\downarrow & & \downarrow \\
O(n) \times O(n')) & \longrightarrow & O(n+n')
\end{array}$$

Therefore, there exists a principal bundle with the subgroup $\mathrm{Spin}(n) \times \mathrm{Spin}(n') \subset O(2n)$ such that

$$V \oplus V' = P \times_{O(2n)} \mathbb{R}^{n+n'}$$

which implies that $V \oplus V'$ is a spinorial vector bundle.

We put

$$\mathrm{Spin}^{\mathbb{C}}(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1) = \mathrm{Spin}(n) \times U(1)/\sim$$

where $(p, z) \sim (-p, -z)$ for all $(p, z) \in \mathrm{Spin}(n) \times U(1)$. It follows that the sequence of groups

$$1 \longrightarrow U(1) \longrightarrow \mathrm{Spin}^{\mathbb{C}}(n) \longrightarrow SO(n) \longrightarrow 1$$

is exact, where $\phi^{\mathbb{C}}(p, z) = \phi(p)$. An oriented vector bundle V of rank n over a compact space X is *a* ${}^{\mathbb{C}}$***spinorial vector bundle*** if there exists a principal bundle P with structure group $\mathrm{Spin}^{\mathbb{C}}(n)$ such that

$$V \cong P \times_{\mathrm{Spin}^{\mathbb{C}}(n)} \mathbb{R}^n.$$

THEOREM 7. *A complex vector bundle (an oriented vector bundle)*

$$V \cong P \times_{U(n)} \mathbb{R}^{2n}$$

of rank n is a ${}^{\mathbb{C}}$spinorial vector bundle, where P is a principal bundle with structure group $U(n)$ and

$$a + bi \longleftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Proof. At first, we suppose the commutative diagram :

$$\begin{array}{ccccc} & & \mathrm{Spin}^{\mathbb{C}}(2n) & & \\ & {}^{\sigma}\nearrow & \copyright & \searrow^{\phi^{\mathbb{C}}} & \\ U(n) & & \longrightarrow & & SO(2n) \end{array}$$

where σ and $\phi^{\mathbb{C}}$ are defined as follows.

For $\alpha \in U(n)$ let $\tau(t)$ be a path in $U(n)$ with $\tau(0) = 1$ and $\tau(1) = \alpha$ ($U(n)$ is arcwise connected). By the exact sequence $(*)$ the map ϕ : $\mathrm{Spin}(2n) \to SO(2n)$ is a convering, and thus there exists a unique path in $\mathrm{Spin}(2n)$ such that $\tilde{\tau}(0) = 1$ and $\phi(\tilde{\tau}(t)) = \tau(t)$. The path $t \mapsto \gamma(t) = \det(\tau(t)) \in U(1) = SO(2)$ may be lifted to $\tilde{\gamma}(t)$ in $\mathrm{Spin}(2) = U(1)$. Then the pair $(\tilde{\tau}(1), \tilde{\gamma}(1)) \in \mathrm{Spin}^{\mathbb{C}}(2n)$ and we define

$$\sigma(\alpha) = (\tilde{\tau}(1), \tilde{\gamma}(1)).$$

In this case, $(\tilde{\tau}(1), \tilde{\gamma}(1))$ is independent of the path $\tau(t)$ connecting α to 1. We shall prove this in the following.

By the exact sequence $(*)$ of groups

$$\phi^{-1}(1) = \{ +1, -1 \},$$
$$\phi^{-1}(\alpha) = \{ +\tilde{\tau}(1), -\tilde{\tau}(1) \},$$
$$\phi^{-1}(\gamma(1)) = \{ +\tilde{\gamma}(1), -\tilde{\gamma}(1) \}.$$

We suppose a path $\tau'(t)$ from α to 1. Then $\phi^{-1}(\tau'(t))$ is a path from $\tilde{\tau}(1)$ to 1 or a path from $-\tilde{\tau}(1)$ to -1 in Spin$(2n)$. Hence, if we take as $\tilde{\tau}'(0) = 1$ $(\in \phi^{-1}(\tau(0)))$, we must have

$$(\tilde{\tau}(1), \tilde{\gamma}(1)) = (\tilde{\tau}'(1), \tilde{\gamma}'(1)) \qquad \text{or} \qquad (\tilde{\tau}(1), \tilde{\gamma}(1)) = (-\tilde{\tau}'(1), -\tilde{\gamma}'(1))$$

and thus σ is wwll-defined. Moreover, σ is homomorphism since $U(n)$ is connected and

$$\sigma(xy) = \sigma(x)\sigma(y) \qquad \forall x, y \in U(n).$$

Thus

$$V \cong P \times_{\mathrm{Spin}^{\mathbb{C}}(2n)} \mathbb{R}^{2n}$$

and V is a ${}^{\mathbb{C}}$spinorial vector bundle.

EXAMPLE 8. There is an example of the homomorphism σ defined as in Theorem 7 as follows. Let $f_1, \cdots, f_n$ be an orthonormal basis of $\mathbb{C}^n$ such that

$$\alpha(f_r) = \exp(i\theta_r) \cdot f_r, \qquad e_{2r-1} = f_r \qquad \text{and} \qquad e_{2r} = i f_r$$

for the corresponding orthonormal basis of $\mathbb{R}^{2n}$. We put

$$S = \prod_{j=1}^{n} \left(\cos \frac{\theta_j}{2} - e_{2j-1} e_{2j} \sin \frac{\theta_j}{2} \right) \exp(\frac{i\theta_j}{2})$$

which is an element of $\mathrm{Spin}^{\mathbf{C}}(2n)$. Then

$$\begin{aligned}\phi^{\mathbf{C}}(S)(f_r) &= Sf_rS^{-1}\\ &= \prod_{j=1}^{n}\left(\cos\frac{\theta_j}{2} - e_{2j-1}e_{2j}\sin\frac{\theta_j}{2}\right) f_r\\ &\quad\times\prod_{j=1}^{n}\left(\cos\frac{\theta_j}{2} + e_{2j-1}e_{2j}\sin\frac{\theta_j}{2}\right)\exp(-\frac{i\theta_j}{2})\\ &= \prod_{j=1}^{n}\left(\cos\frac{\theta_j}{2} - e_{2j-1}e_{2j}\sin\frac{\theta_j}{2}\right) f_r\\ &\quad\prod_{j=1}^{n}\left(\cos\frac{\theta_j}{2} + e_{2j-1}e_{2j}\sin\frac{\theta_j}{2}\right)\end{aligned}$$

Since for $j \neq r$

$$\left(\cos\frac{\theta_j}{2} - e_{2j-1}e_{2j}\sin\frac{\theta_j}{2}\right)\cdot\left(\cos\frac{\theta_j}{2} + e_{2j-1}e_{2j}\sin\frac{\theta_j}{2}\right) = 1$$

and

$$\begin{aligned}&\left(\cos\frac{\theta_r}{2} - e_{2r-1}e_{2r}\sin\frac{\theta_r}{2}\right) f_r \left(\cos\frac{\theta_r}{2} + e_{2r-1}e_{2r}\sin\frac{\theta_r}{2}\right)\\ &\qquad = \left(\cos\frac{\theta_r}{2} + i\,\sin\frac{\theta_r}{2}\right)^2 e_{2r-1} = \exp(i\theta_r)\cdot f_r\end{aligned}$$

we have

$$Sf_rS^{-1} = \exp(i\theta_r)\cdot f_r = \alpha(f_r).$$

Therefore

$$\theta(\alpha) = \prod_{j=1}^{n}\left(\cos\frac{\theta_j}{2} - e_{2j-1}e_{2j}\sin\frac{\theta_j}{2}\right)\exp(\frac{i\theta_j}{2}).$$

References

1. O. Husemoller, *Fibre Bundles*, Springer-Verlag, 1975.
2. M. Karoubi, *K-Theory (An Introduction)*, Springer-Verlag, 1978.
3. H. Lee and K. Lee, *A Note on Spinorial Structures in Vector Bundles*, Honam Math. J., Vol. 11 #1 (1989), 7–14.
4. ———, *On the Orientations and Spinorial Structures on Vector Bundles*, Comm. Korean Math. Soc., Vol. 5 #1 (1990), 29–35.
5. ———, *Sheaves, Complex Manifolds, Index Theorem*, Hyonseol Publ. Co., 1984.
6. K. Lee, *Foundations of Topology, Vol. 1 and Vol. 2*, Hakmun Publ. Co., 1984.