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TAIL EVENTS FOR RANDOM WALKS WITH TIME STATIONARY RANDOM DISTRIBUTION FUNCTION

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1. Introduction

Let $\mathcal{F}$ be a set of distributions on $\Re^1$ with the topology of weak convergence, and let $\mathcal{A}$ be the σ–field generated by the open sets. We denote by $\mathcal{F}_1^\infty$ the space consisting of all infinite sequence $(F_1, F_2, \dots), F_n \in \mathcal{F}$, for all $n \geq 1$ and $\Re_1^\infty$ the space consisting of all infinite sequences $(x_1, x_2, \dots)$ of real numbers. Take the σ–field $\mathcal{A}_1^\infty$ to be the smallest σ–field of subsets of $\mathcal{F}_1^\infty$ containing all finite–dimensional rectangles and take $\mathcal{B}_1^\infty$ to be the Borel σ–field of $\Re_1^\infty$. Let $\omega = (F_1^\omega, F_2^\omega, \dots)$ be the coordinate process in $\mathcal{F}_1^\infty$ and ν its distribution on $\mathcal{A}_1^\infty$. Let θ be the coordinate shift : $\theta^k(\omega) = \omega'$ with $F_n^{\omega'} = F_{n+k}^\omega, k = 1, 2, \dots$. On $(\Re_1^\infty, \mathcal{B}_1^\infty)$ we also define the shift transformation $\sigma : \Re_1^\infty \to \Re_1^\infty$ by $\sigma(x_1, x_2, \dots) = (x_2, x_3, \dots)$. ν is called stationary if for every $A \in \mathcal{A}_1^\infty$, $\nu(\theta^{-1}(A)) = \nu(A)$ and let π be its marginal distribution. Let $\mathcal{J}$ be the σ–field of invariant sets in $\mathcal{B}_1^\infty$, that is, $\mathcal{J} = \{B | \sigma^{-1}(B) = B, B \in \mathcal{B}_1^\infty\}$. For each ω define a probability measure P_ω on $(\Re_1^\infty, \mathcal{B}_1^\infty)$ so that $P_\omega = \Pi_{i=1}^\infty \mathcal{F}_i^\omega$. Define the process $\{X_n\}$ on $(\Re_1^\infty, \mathcal{B}_1^\infty)$ such that $X_n(x_1, x_2, \dots) = x_n$ and set $S_n = X_1 + X_2 + \cdots + X_n$. By the definition of $P_\omega, \{X_n\}$ are independent with respect to P_ω and we also note that $\{X_n\}$ is a sequence of independent and identically distributed random variables when $\mathcal{F}$ has just one element.

Consider a sequence $\{h_n(x)\}$ of real valued measurable functions satisfying the system of equations

(1) $h_n(x) = \int_{-\infty}^\infty h_{n+1}(x+y) dF_{n+1}^\omega(y), n = 1, 2, \dots$.

We shall consider only positive bounded solutions, i.e., we impose

(2) $0 \leq \inf_{n,x} h_n(x), \sup_{n,x} h_n(x) < \infty$.

We also impose the requirement of continuity, that is,

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(3) $h_n(x)$ is continuous, $n = 1, 2, \ldots$.

The set of all sequences $h(h = \{h_n(x)\})$ satisfying (1),(2) and (3) we denote by $\mathcal{H}^\omega$. A sequence of functions $\{h_n(x)\}$ is trivial if there exists a constant c such that $h_n(x) = c$ for all n and all x ; $\mathcal{H}^\omega$ is trivial if all $h \in \mathcal{H}^\omega$ are trivial. In the identically distributed case, $F_1^\omega = F_2^\omega = \cdots$, it is well known that if F_1^ω does not have a jump of 1 only two cases can occur ; (i) $\mathcal{H}^\omega$ is trivial, (ii) $\mathcal{H}^\omega$ is not trivial but for some $d > 0, h_n(x) = h_n(x - d)$ for all n and x for each $h \in \mathcal{H}^\omega$(arithmetic case) ; only if for some number $x_0, F_1^\omega(x) = 0$ for $x < x_0, F_1^\omega(x) = 1$ for $x > x_0$ will one have (iii) non-periodic h present in $\mathcal{H}^\omega$ (degenerate case). Our concern is to extend the above to our model.

In 1966, Orey [3] wrote a paper with this topic for the non-identically distributed case. This paper is based on that paper.

2. Bounded Case

In this section we take our distribution functions to be uniformly bounded. We shall use the condition

$$\text{(A)} \qquad \int_{-N}^{N} dF(x) = 1 \quad \text{for all} \quad F \in \mathcal{F},$$

where N is a real number.

A sequence of real numbers $\{b_n\}$ is a likely sequence if for every $\epsilon > 0$, $\inf P[|X_n - b_n| < \epsilon] > 0$.

LEMMA 1. (A) *implies the existence of a likely sequence.*

Proof. See Lemma 2.1 [3].

We also use the following condition.

$$\text{(B)} \qquad \int_{(-\epsilon,\epsilon)} dF(x) > 0 \text{ for all } F \in \mathcal{F}, \text{ for every } \epsilon > 0.$$

We introduce the following notation :

$$\Gamma_\omega = \{x | \sum_{k=1}^{\infty} P_\omega\{|X_k - x| < \epsilon\} = \infty \quad \text{for every} \quad \epsilon > 0\},$$

$$H = \{y | \iint_{B(y,\epsilon)} dF(x)\pi(dF) > 0 \quad \text{for every} \quad \epsilon > 0\},$$

where $B(y,\epsilon) = (y-\epsilon, y+\epsilon)$.

Let A^* denote the closure of the group generated by A. Γ^*_ω and H^*, being closed subgroups of the reals, $\Re$, must either be $\Re$, consist of 0 only, or be generated by some positive number respectively. $\mathcal{F}$ is trivial if $\pi\{F|F \text{ is degenerate}\} = 1$.

LEMMA 2. *Let $\mathcal{F}$ be not trivial, then under* (A) *and* (B) $H^* \neq \{0\}$.

Proof. Suppose $H^* = \{0\}$, then H should be $\{0\}$. Then for every $x(\neq 0)$, there exist $\epsilon(x)$ depending on x such that $\iint_{B(x,\epsilon(x))} dF(x)\pi(dF) = 0$. Since for every $\delta > 0, [-N,N] - B(0,\delta)$ is compact, we have

$$\iint_{[-N,N]-B(0,\delta)} dF(y)\pi(dF) = 0.$$

Since δ is arbitrary, we have

$$\iint_{[-N,N]-\{0\}} dF(x)\pi(dF) = 0, \quad \text{that is,} \quad \pi\{F| \int_{\{0\}} dF(x) = 1\} = 1.$$

This contradicts that $\mathcal{F}$ is not trivial. This proved lemma.

THEOREM 1. *Suppose $\mathcal{F}$ is not trivial and ν is stationary ergodic. Then under* (A) *and* (B), *there exist two possibilities :*

(i) $H^* = \Re$,

(ii) $H^* = \{nd \mid n = 0, \pm 1, \dots\}$ *for some* $d > 0$.

Condition (i) *implies*

$$\nu\{\omega|\mathcal{H}^\omega \text{ is trivial }\} = 1.$$

Condition (ii) *implies*

$$\nu\{\omega| \text{ every } h \in \mathcal{H}^\omega \text{ has period } d\} = 1 \text{ and}$$

$$\pi\{F| \int_{\{nd \mid n=0,\pm 1,\dots\}} dF(x) = 1\} = 1.$$

Proof. By Lemma 2, H^* is either $\Re$ or $\{nd \mid n = 0, \pm 1, \dots\}$ for some $d > 0$. Now suppose $H^* = \Re$. Then there exists countable subset $D \subset H$ such that $D^* = \Re$. If $x \in D$, then by the ergodic theorem

$$\frac{1}{n}\sum_{k=1}^{n} \int_{B(x,\epsilon)} dF_k^\omega(x) \to \iint_{B(x,\epsilon)} dF(x)\pi(dF) > 0, \tag{4}$$

for every $\epsilon > 0, \nu -$ a.e. ω. Then $x \in \Gamma_\omega, \nu$–a.e. ω, and hence $D \subset \Gamma_\omega \nu$–a.e., ω since D is countable. So $\Re = D^* \subset \Gamma^*_\omega \subset \Re$ i.e., $\Gamma^*_\omega = \Re, \nu$–a.e., ω. By Theorem 3.1 [3] $\mathcal{H}^\omega$ is trivial ν–a.e., ω.

Next suppose $H^* = \{nd \,|\, n = 0, \pm 1, \dots\}$ for some $d > 0$. Using the same argument as in Lemma 2, we have

$$(5) \qquad \pi\{F| \int_{\{nd \,|\, n=0,\pm1,\dots\}} dF(x) = 1\} = 1.$$

And using (4) we can easily prove that

$$\nu\{\omega|\Gamma^*_\omega = \{nd \,|\, n = 0, \pm 1, \dots\}\} = 1.$$

Clearly $\sum_{n=1}^{\infty} X_n \bmod d = 0$ with respect to P_ω ν–a.e. ω by (5). Hence by Theorem 3.1 [3] $\nu\{\omega|$ every $h \in \mathcal{H}^\omega$ has period $d\} = 1$, for some d. This completes proof.

It is noted that the absence of non–trivial solutions of (1) and (2) is equivalent to the zero–one law for the $\{S_n\}$ process (i.e., the tail σ–field of $\{S_n\}$ contains only sets of probability one or zero), and hence we have the following result immediately.

COROLLARY 1. *Suppose every $F \in \mathcal{F}$ is distribution function on Z, the set of integers. Then under* (A) *and* (B), *we have*

$$\nu\{\omega|P_\omega(A) \in \{0,1\}\} = 1 \quad \nu - \text{a.e.} \quad \omega$$

where A is any tail event of $\{S_n\}$.

3. Countable Case

In this section we shall use the following condition

(C) $\quad \mathcal{F} = \{F_n|n = 1, 2, \dots\}, \quad \pi\{F_n\} > 0$ for $n = 1, 2, \dots$.

We introduce the following notation :

$$\text{Supp}(F) = \{y| \int_{B(y,\epsilon)} dF(x) > 0, \text{ for every } \epsilon > 0\},$$

$$H_F = \{y| \int_{B(y,\epsilon)-y'} dF(x) > 0, \text{ for every } \epsilon > 0\},$$

where y' is some fixed element in Supp(F).

Note that H^*_F is independent of choice of y' since $H_F \subset \text{Supp}(F) - \text{Supp}(F) \subset H^*_F$. Now we use the following definition.

DEFINITION 1.

$$dF = \begin{cases} 0 & \text{if } H_F^* = \Re, \\ d & \text{if } H_F^* = \{nd \mid d = 0, \pm 1, \dots\}, \\ \infty & \text{if } H_F^* = \{0\}. \end{cases}$$

We note that if $\omega = (F_1^\omega, F_2^\omega, \dots), F_n^\omega = F$ for all $n \geq 1$, every $h \in \mathcal{H}^\omega$ has period dF. For this see the first part of the proof of Theorem 3.1 [3].

LEMMA 3. *Suppose ν is stationary ergodic, then under* (C)

$$\nu\{\omega \mid \text{ every } h \in \mathcal{H}^\omega \text{ has period } d\} = 1,$$

where $d = dF$ for all $F \in \mathcal{F}$.

Proof. By ergodic theorem, we have

$$\frac{1}{n} \sum_{k=1}^{n} 1_{\{F\} \times \mathcal{F} \times \mathcal{F} \times \cdots} \theta^k(\omega) \to \pi\{F\} > 0 \quad \nu - \text{a.e.} \quad \omega.$$

This means $\nu\{\omega | F_n^\omega = F \text{ i.o. }\} = 1$ for all $F \in \mathcal{F}$, and Corollary 1 [3] applies.

THEOREM 2. *Suppose ν is stationary ergodic and $\mathcal{F}$ is not trivial. Then under* (C), *there are two possibilities*

(i) $(\cup_{n=1}^{\infty} H_{F_n}^*)^* = \Re$,
(ii) $(\cup_{n=1}^{\infty} H_{F_n}^*)^* = \{md | m = 0, \pm 1, \dots\}$ *for some* $d > 0$.

Condition (i) *implies*

$$\nu\{\omega | \mathcal{H}^\omega \text{ is trivial }\} = 1.$$

Condition (ii) *implies*

$$\nu\{\omega \mid \text{ every } h \in \mathcal{H}^\omega \text{ has period } d\} = 1.$$

Proof. Suppose $\mathcal{F}$ is not trivial, so $(\cup_{n=1}^{\infty} H_{F_n}^*)^* \neq \{0\}$. Hence $(\cup_{n=1}^{\infty} H_{F_n}^*)^*$ is either $\Re$ or $\{nd \mid n = 0, \pm 1, \dots\}$ for some $d > 0$. First suppose

$(\cup_{n=1}^{\infty} H_{F_n}^*)^* = \Re$. Then for every $\epsilon > 0$, there exist $dF_{n1}, \ldots, dF_{nk}$ and $m_1, \ldots, m_k$ such that $|m_1 dF_{n1} + \cdots + m_k dF_{nk}| < \epsilon$ where $m_i \in Z \;\; i = 1, 2, \ldots, k$. We know that by Lemma 3 that $\nu\{\omega|$ every $h \in \mathcal{H}^\omega$ has period dF_{ni} $i = 1, 2, \ldots, k\} = 1$ and hence

$$\nu\{\omega| \text{ every } h \in \mathcal{H}^\omega \text{ has period } |m_1 dF_{n1} + \cdots + m_k dF_{nk}|(< \epsilon)\} = 1.$$

Since ϵ is arbitrary, we have

$$\nu\{\omega|\mathcal{H}^\omega \text{ is trivial}\} = 1.$$

Next suppose $(\cup_{n=1}^{\infty} H_{F_n}^*)^* = \{nd \,|\, n = 0, \pm1, \pm2, \ldots\}$ for some $d > 0$.

Then there exist $dF_{n1}, \ldots, dF_{nk}$ and $(m_1, \ldots, m_k) \in Z^k$ such that $m_1 dF_{n1} + \cdots + m_k dF_{nk} = d$. By similar argument as above, we have $\nu\{\omega|$ every $h \in \mathcal{H}^\omega$ has period $d\} = 1$.

LEMMA 4. *Suppose* $\cap_{n=1}^{\infty} Supp(F_n) \neq \phi$ *and suppose that* ν *is stationary ergodic and* $\mathcal{F}$ *is not trivial. Then under the conditions of Theorem 2 condition* (ii) *in Theorem 2 implies* $\pi\{F| \int_{\{nd+x|n=0,\pm1,\ldots\}} dF(y) = 1\} = 1$, *for some* $x \in \cap_{n=1}^{\infty} Supp(F)$.

Proof. Take $x \in \cap_{n=1}^{\infty} \mathrm{Supp}(F_n)$ and note that $H_{F_n}^* + x = (\mathrm{Supp}(F_n))^*$. Since $H_{F_n}^* \subset \{nd \,|\, n = 0, \pm1, \ldots\}$ by Theorem 2 (ii), $\mathrm{Supp}(F_n) \subset \{nd + x|n = 0, \pm1, \ldots\}$ for $n = 1, 2, \ldots$. This completes Lemma.

The following result is immediate from Theorem 2 and Lemma 4.

COROLLARY 2. *Suppose all* F_n *is distribution function of* Z *and* $\cap_{n=1}^{\infty} SuppF_n \neq \phi$. *Then under the condition of Theorem 2,* $\nu\{\omega|P_\omega(A) \in \{0,1\}\} = 1$ *where* A *is any tail event for* $\{S_n\}$.

4. General Case

When the X_n are not uniformly bounded but there exist a likely sequence, it is still true that for every $y \in \Gamma_\omega^*$, each $h \in \mathcal{H}^\omega$ will have period d : this follows from the first part of the proof of Theorem 3.1 [3]. Using this fact we can prove the following results in a similarly way. So we give the results without proofs.

THEOREM 3. *Suppose there exists x such that* $\inf_{F\in\mathcal{F}} \int_{B(x,\epsilon)} dF(x) > 0$. *Suppose ν is stationary ergodic and $\mathcal{F}$ is not trivial. Then there exist two possibilities :*

(i) $(H-x)^* = \Re$,

(ii) $(H-x)^* = \{nd \mid n = 0, \pm 1, \dots\}$ *for some* $d > 0$.

Condition (i) *implies*

$$\nu\{\omega|\mathcal{H}^* \text{ is trivial }\} = 1.$$

Condition (ii) *implies*

$$\nu\{\omega| \text{ every } h \in \mathcal{H}^* \text{has periodd}\} = 1 \text{ and }.$$

$$\pi\{F| \int_{\{x+nd \mid n=0,\pm1,\dots\}} dF(x) = 1\} = 1.$$

COROLLARY 3. *Let $\mathcal{F}$ be a set of distribution functions on Z. Then under the conditions of Theorem 3, we have* $\nu\{\omega|P_\omega(A) \in \{0,1\}\} = 1$ *where A is any tail event of* $\{S_n\}$.

The following theorem is immediate consequence of Corollary 1 [3] and the ergodic theorem.

THEOREM 4. *Suppose there exist $F \in \mathcal{F}$ such that $\pi\{F\} > 0$ and $H_F^* = \Re$ and ν is stationary ergodic, then*

$$\nu\{\omega|\mathcal{H}^\omega \text{ is trivial}\} = 1.$$

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