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TIME CHANGED STOCHASTIC INTEGRALS

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1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t)_{t\geq 0}$ be a reference family. We introduce the following notations;

$\mathcal{M}^2$= the family of all continuous locally square integrable martingales $M(t)$ and $M(0) = 0$ a.s.

$\mathcal{A}_1$ = the family of all continuous adapted process $A(t)$ and $A(0) = 0$, $t \mapsto A(t)$ is non-decreasing a.s.

$\mathcal{A}_2$ = the family of all continuous adapted process $A(t)$ and $A(0) = 0$, $t \mapsto A(t)$ is of bounded variation on finite interval a.s.

$\mathcal{L}^2$ = the family of all real predictable process $\Phi(t)$ such that there exists a sequence of stopping times σ_n such that $\sigma_n \uparrow \infty$ a.s. and

$$E\left[\int_0^{T\wedge\sigma_n} \Phi^2(t,\omega)d\langle M\rangle(t)\right] < \infty$$

for every $T > 0, n = 1, 2, \cdots$ where $\langle M\rangle(t)$ is a quadratic variation process of $M(t) \in \mathcal{M}^2$. ([2, II. Definition 2.1], [4, IV.26])

$\mathcal{Q}$ = the family of all continuous semimartingales $X(t)$. ([2, III. Definition 1.1], [3])

By a time changed process we mean any process $\eta(t) \in \mathcal{A}_1$ such that, with probability 1, $t \mapsto \eta(t)$ is a stopping time and $\lim_{t\uparrow\infty} \eta(t) = \infty$.

In this note, we define the time changed reference family and represent the stochastic processes with respect to one reference family as the time changed stochastic integrals with respect to a process on the other reference family.

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2. The Main Results

We begin with:

LEMMA 1. *Let $M(t) \in \mathcal{M}^2$ such that $\lim_{t\uparrow\infty}\langle M\rangle(t) = \infty$ a.s. If we set*

$$\tau_t = inf\{u : \langle M\rangle(u) > t\}$$

and $\widetilde{\mathcal{F}}_t = \mathcal{F}_{\tau_t}$, then the time changed process $B(t) = M(\tau_t)$ is an $(\widetilde{\mathcal{F}}_t)$-Brownian motion.

Proof. See Ikeda and Watanabe [2, II.Theorem 7.2].

Let $M(t) \in \mathcal{M}^2$ be an $(\mathcal{F}_t)$-well measurable process and $\eta(t)$ a process of time change. Define the mapping

$$(T^{\eta}M)(t) = M(\tau_t).$$

Then the mapping T^{η} is an $(\widetilde{\mathcal{F}}_t)$-well measurable process.

We are ready for Representation theorem:

THEOREM 2. *Suppose that $\lim_{t\uparrow\infty}\langle M\rangle(t) = \infty$ a.s. Then $\eta(t) = \langle M\rangle(t)$ is a process of time change and there exists $\Phi(t) \in \mathcal{L}^2$ such that*

$$M(t) = \int_0^{\tau_t} \Phi(s)dB(s) \tag{1}$$

where $B(t)$ is an $(\widetilde{\mathcal{F}}_t)$-Brownian motion.
In other words, every martingales with respect to $(\mathcal{F}_t)$ can be represented as the time changed stochastic integrals with respect to Brownian motions on $(\widetilde{\mathcal{F}}_t)$.

Proof. Let $\tau_t = inf\{u : \langle M\rangle(u) > t\}$. Then we have

$$\{\langle M\rangle(t) \le u\} = \{t \le \tau_u\} \in \widetilde{\mathcal{F}}_u$$

and

$$\{\tau_t \le u\} = \{t \le \langle M\rangle(u)\} \in \mathcal{F}_u.$$

Hence, $\eta(t)$ is an $(\widetilde{\mathcal{F}}_t)$-stopping time and τ_t is an $(\mathcal{F}_t)$-stopping time. From the definition of stochastic integral, there exists $\phi(t) \in \mathcal{L}^2$ such that

$$M(t) = \int_0^t \phi(s)dB_{\phi}(s)$$

where $B_\phi(t)$ is an $(\mathcal{F}_t)$-Brownian motions.
Therefore, we have

$$(T^\eta M)(t) = \int_0^{\tau_t} \phi(s)dB_\phi(s)$$

and letting $\Phi(t) = \phi(\eta(t))$ and $B(t) = B_\phi(\eta(t))$, this result follows.

In the proof of Theorem 2, we know that every Brownian motion with respect to $(\widetilde{\mathcal{F}}_t)$ can be represented as the time changed stochastic integrals with respect to basic martingales on $(\mathcal{F}_t)$.

Let $X(t)$, $Y(t) \in \mathcal{Q}$. That is, $X(t)$ and $Y(t)$ are decomposed in the Canonical form

$$X(t) = X(0) + M_X(t) + A_X(t), \;\; Y(t) = Y(0) + M_Y(t) + A_Y(t)$$

where $X(0)$, $Y(0)$ is $\mathcal{F}_0$-measurable, and $M_X(t)$, $M_Y(t) \in \mathcal{M}^2$, and $A_X(t)$, $A_Y(t) \in \mathcal{A}_2$. ([2, III. Definition 1.1], [3], [4, IV.31]) Denote the quadratic covariation process of $X(t)$ and $Y(t)$ by $\langle X, Y\rangle(t)$.([2, II. Definition 2.1], [4, IV 26])

We define *the symmetric $\mathcal{Q}$- multiplication* ([2, p.100])

$$Y \circ dX = Y \cdot dX + \frac{1}{2}d\langle M_X, M_Y\rangle \text{ for } X(t), Y(t) \in \mathcal{Q}$$

and *Stratonovich integral* or *Fisk integral* ([4, IV 46])

$$S_t = \int_0^t Y \circ dX = \int_0^t Y \cdot dX + \frac{1}{2}\langle X, Y\rangle.$$

We now meet:

THEOREM 3. *Let $X(t)$, $Y(t) \in \mathcal{Q}$. Then there exist $\Phi(t), \Psi(t) \in \mathcal{L}^2$ such that*

$$(2)^* \qquad \langle X, Y\rangle(t) = \int_0^{\tau_t} \Phi(s)\Psi(s)d\langle B_X, B_Y\rangle(s)$$

where $B_X(t)$ and $B_Y(t)$ are $(\widetilde{\mathcal{F}}_t)$ -Brownian motions corresponding with $X(t)$ and $Y(t)$, respectively. Moreover,

$$E\,|\langle X,Y\rangle(t)| < \infty \quad \textit{for} \quad \textit{all} \quad t$$

In other words, every quadratic covariation of semimartingales with respect to $(\mathcal{F}_t)$ can be represented as the time changed stochastic integrals with respect to quadratic covariation of basic martingales on $(\widetilde{\mathcal{F}}_t)$.

Proof. Operating Itô and Stratonovich differential to S_t, respectively, we obtain

$$\begin{aligned} dS_t &= Y \cdot dX + \frac{1}{2} d\langle X,Y\rangle \\ &= Y \circ dX = Y \cdot dX + \frac{1}{2} d\langle M_X, M_Y\rangle. \end{aligned}$$

Therefore, we have $d\langle X,Y\rangle = d\langle M_X, M_Y\rangle$ and operating the Itô differential, we obtain

$$\langle X,Y\rangle(t) = \langle M_X, M_Y\rangle(t).$$

From the Representation (1), we know that there exist $\Phi(t), \Psi(t) \in \mathcal{L}^2$ such that

$$\langle X,Y\rangle(t) = \left\langle \int_0^{\tau_t} \Phi(s) dB_X(s), \int_0^{\tau_t} \Psi(s) dB_Y(s) \right\rangle$$

where $B_X = B_X(\langle M\rangle(t))$ and $B_Y = B_Y(\langle M\rangle(t))$ are $(\widetilde{\mathcal{F}}_t)$-Brownian motions by Lemma 1.
Since

$$E\left[\left(\int_{\tau_s}^{\tau_t} \Phi(s) dB_X(s)\right)\left(\int_{\tau_s}^{\tau_t} \Psi(s) dB_Y(s)\right) \mid \widetilde{\mathcal{F}}_s\right]$$

$$E\left[\int_{\tau_s}^{\tau_t} \Phi(u)\Psi(u) d\langle B_X, B_Y\rangle(u) \mid \widetilde{\mathcal{F}}_s\right],$$

it follows that there exist $\Phi(t)$, $\Psi(t) \in \mathcal{L}^2$ such that

$$\langle X,Y\rangle(t) = \int_0^{\tau_t} \Phi(s)\Psi(s) d\langle B_X, B_Y\rangle(s).$$

*The notation $\int$ is a Itô integral

On the other hand, since we have the inequality

$$\int_0^{\tau_t} \Phi(s)\Psi(s)d\,|\langle B_X, B_Y\rangle|\,(s)$$
$$\leq \left\{\int_0^{\tau_t} \Phi^2(s)d\langle B_X\rangle(s)\right\}^{\frac{1}{2}} \left\{\int_0^{\tau_t} \Psi^2(s)d\langle B_Y\rangle(s)\right\}^{\frac{1}{2}},$$

if $\Phi(t) \in \mathcal{L}^2$, $\Psi(t) \in \mathcal{L}^2$, then

$$E\,|\langle X, Y\rangle(t)| < \infty \quad \text{for} \quad \text{all} \quad t.$$

We conclude with:

COROLLARY 4. *If $B_X(t) = B_Y(t)$ for all t, then Representation* (2) *becomes*

$$(3)^* \qquad \langle X, Y\rangle(t) = \int_0^{\tau_t} \Phi(s)\Psi(s)ds$$

Proof. Since the fact that continuous process $B(t)$ is a Brownian motion is equivalent that both $t \mapsto B(t)$ and $t \mapsto B(t)^2 - t$ are martingales ([1, VII. Theorem 11.9]), we have

$$\langle B_X, B_Y\rangle(t) = t.$$

References

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*The notation $\int$ is a Riemann integral