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ON DIFFERENTIABLE SEMIGROUPS

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1. Preliminaries

The notion of a semigroup with differentiable multiplication based on an ordinary differentiable manifold was studied in [3], [4]. This conception eliminates many interesting cases. For example, the Heisenberg group G of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

such that x, y, z are real. This group may be identified with $\mathbb{R}^3$ (as a manifold). Let S be the subsemigroup of G defined by $x, y \geq 0$ and $0 \leq z \leq xy$. When identified with a subset of $\mathbb{R}^3$, S is the region in the first octant below the graph of $z = xy$. Thus S has a cusp at the identity. This example illustrates the need' to allow corners, cusps, and possible other irregularities in the boundary. Thus we need a generalized differentiable manifold.

The following definitions in this paper are due to Graham in [1]. A subset A of a topological space X is said to be *admissible* set if A has dense interior in X. Let $A \subseteq E$ be an admissible set of a Banach space E, F be a Banach space and let $f : A \to F$ be a function. A linear map $T \in L(E, F)$, the Banach space of continuous linear maps from E to F, is a *strong derivative* of f at $a \in A$ if for each $\varepsilon > 0$, there is a $\delta > 0$ such that if each of x and y is within δ of a then $|f(y) - f(x) - T(y - x)| < \varepsilon |y - x|$. We denote T by $df(a)$. As usual, f is C^1_s if $df(x)$ exists for each $x \in A$. Inductively, f is C^k_s if f is C^1_s and df is C^{k-1}_s. Finally, f is C^∞_s if f is C^k_s for all positive integers k. If f is C^k_s, then the j^{th} *strong derivative* of f $(j \leq k)$ is the map $d^j f = d(d^{j-1}) : A \to L_j(E, F)$, the Banach space of continuous j-multilinear maps E^j to F.

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Let E be a Banach space. An *E-manifold* (with generalized boundary) is a regular topological space M such that for each $p \in M$ there is an open set $U \subset M$ about p and a homeomorphism φ from U onto an admissible subset of E. An *C_s^k atlas* for an E-manifold M is a collection $\mathcal{A}$ of functions satisfying (i) each $\varphi \in \mathcal{A}$ is a homeomorphism from an open subset $dom\varphi$ of M onto an admissible subset $im\varphi$ of E, (ii) $M = \bigcup dom\varphi(\varphi \in \mathcal{A})$ and (iii) $\psi \circ \varphi^{-1}$ is a C_s^k map for each $\psi, \varphi \in \mathcal{A}$. A *$C_s^k$ manifold* is a pair $(M, \mathcal{D})$ where M is an E-manifold and $\mathcal{D}$ is a maximal C_s^k atlas. Let M be a C_s^k E- manifold and let N be a C_s^k F-manifold. For each chart φ on M and ψ on N, define $\varphi \times \psi$ by $(\varphi \times \psi)(p,q) = (\varphi(p), \psi(q)) \in E \times F$. Then $dom(\varphi \times \psi)$ is open in $M \times N$ with the product topology and $im(\varphi \times \psi)$ is an admissible subset of $E \times F$. It follows that the collection of all such maps $\varphi \times \psi$ is C_s^k atlas for $M \times N(E \times F$-manifold). The C_s^k differentiable structure generated by this atlas is called the *product structure* for $M \times N$. In similar way, one may show that any finite Cartesian product of C_s^k manifolds is a C_s^k manifold.

Let M be a C_s^k E-manifold and let $p \in M$. If φ and ψ are charts at p $(\varphi, \psi \in \mathcal{D})$ and if $v, w \in E$, then (φ, v) is *p-equivalent* to (ψ, w) if $d(\psi \circ \varphi^{-1})(\varphi(p))v = w$. Clearly, this gives an equivalence relation on $\mathcal{A} \times E$. Let T_pM denote the set of equivalence classes $[(\varphi, v)]_p$ where φ is a chart at p and $v \in E$. The *tangent space* of M at p is the set T_pM with the unique vector space structure such that $\hat{\varphi}_p : E \to T_pM$ defined by $\hat{\varphi}_p(v) = [(\varphi, v)]_p$ is isomorphism for each chart φ at p. Let M and N be C_s^k manifolds and let $f : M \to N$ be a C_s^k map. If $p \in M$, then the *(strong) derivative* of f at p is the map $df(p) : T_pM \to T_{f(p)}N$ defined by $df(p) = \hat{\psi}_{f(p)} \circ d(\psi \circ f \circ \varphi^{-1})(\varphi(p)) \circ (\hat{\varphi}_p)^{-1}$, where φ is a chart at p and ψ is a chart at $f(p)$. The definition of $df(p)$ is independent of the choice of charts φ and ψ. Let $TM = \{(p, v) | p \in M, v \in T_pM\}$. For each chart $\varphi : U \subset M \to A \subset E$, define $T\varphi$ from $TU = \{(p, v) | p \in U, v \in T_pM\}$ onto $A \times E$ by $T\varphi(p, v) = (\varphi(p), d\varphi(p)v) = (\varphi(p), (\hat{\varphi}_p)^{-1}v)$. The collection of sets of the form $(T\varphi)^{-1}(W)$, where φ is a chart on M and W is an open subset of $E \times E$, is a base for a topology on TM. With this topology, the collection of all maps $T\varphi$ is a C_s^{k-1} atlas for TM as an $E \times E$-manifold. The *tangent bundle* of M is the C_s^k map $\pi : TM \to M$ defined by $\pi(p, v) = p$. A *vector field* on M is a section of π, that is, a map $X : M \to TM$ such that $\pi \circ X = 1_M$. Given a vector field X on M, we denote X_p by the image $X(p)$ for each $p \in M$.

2. C_s^k semigroup and its Lie algebra

A semigroup S with multiplication m is a C_s^k *semigroup* if S itself is a C_s^k manifold and m is a C_s^k map from $S \times S$ to S, where $S \times S$ carries the product structure. A C_s^k semigroup S is locally compact (compact) if S is locally compact (compact) topological space. A vector field X on a C_s^k semigroup S is *right-invariant* if $d\rho_b(a)(X_a) = X_{ab}$ for each $a, b \in S$. The collection of right-invariant vector fields of S is denoted by $L(S)$. Let S be a C_s^k semigroup with identity (C_s^k monoid). Then $L(S)$ is linearly isomorphic to the tangent space T_1S. Since T_1S can be given the structure of a Banach space, $L(S)$ may be given the structure of a Banach space. If $k \geq 3$, then $L(S)$ is a Lie algebra under Lie bracket of vector fields [1].

Remark. For a locally compact C_s^∞ monoid S, let $W(S) = \{X \in L(S) | X_1 = \alpha'(0)$ for some one-parameter submonoid of $S\}$. Define $\exp : W(S) \to S$ by $\exp(X) = \alpha(1)$ where $\alpha'(0) = X_1$. Then $W(S)$ is a closed wedge in $L(S)$ and exp is a C_s^∞ diffeomorphism into S on a neighborhood of 0 (II. Corollary 6.4. from [1]).

Let G be a Lie group with its Lie algebra $L(G)$. Then there is a one-to-one correspondence between $L(G)$ and the set of all one-parameter subgroups of G. For C_s^∞ monoids, we may have the following result similar with Lie group. Throughout we will denote with $\mathbb{R}^+$ the set of non-negative reals.

THEOREM 1. *Let S be a locally compact C_s^∞ monoid. Then there is one-to-one correspondence between the set $W(S)$ and the set of all one-parameter submonoids of S.*

Proof. For each $X \in W(S)$, there is a one-parameter submonoid α such that $\alpha'(0) = X_1$. If β is any other one-parameter submonoid of S such that $\beta'(0) = X_1$, then α, β are integral curve of X satisfying $\alpha(0) = \beta(0)$. Thus $\alpha = \beta$. For $X, Y \in W(S)$, if $\exp(tX) = \exp(tY)$ for all $t \in \mathbb{R}^+$, then $X_1 = Y_1$ and so $X = Y$ since $L(S)$ is isomorphic to T_1S under $X \mapsto X_1$. If α is a one-parameter submonoid of S, then α is a C_s^∞ map ([1], II. Corollary 6.2). Hence $\alpha'(0) = d\alpha(0)(1) \in T_1S$ and there is a unique $X \in L(S)$ such that $X_1 = \alpha'(0)$. Thus $X \in W(S)$ and $\alpha(t) = \exp(tX)$ for all $t \in \mathbb{R}^+$.

Note. Let $S = \{(x, y)|x > 0, y \in (0, x^2)\}$ together with $(0,0)$. Then S is a C_s^∞ submonoid of the additive plane and $L(S) = (\mathbb{R}^2, +)$. Note that the only one-parameter submonoid of S is the trivial one. Hence $W(S) = \{0\}$.

If L is a finite dimensional Lie algebra, then all norms on L which make addition continuous are equivalent and in this case there is a norm such that $|[X, Y]| \leq |X||Y|$ for all $X, Y \in L$.

Let B be an open ε-ball around 0 with respect to this norm such that (i) $(X, Y) \to X * Y = X + Y + \frac{1}{2}[X, Y] + \cdots$ is defined and continuous on $B \times B$, where * is the multiplication given by the absolutely convergent Campbell-Hausdorff series, and (ii) all triple products are defined and associative. Such neighborhood always exists. We assume that B is fixed in this section. We say that $S \subseteq B$ is a ***local semigroup*** with respect to B if $0 \in S$ and $(S * S) \cap B \subseteq S$. For a local semigroups, we set $\mathbb{L}(S) = \{X \in L|\mathbb{R}^+ X \cap B \subseteq \bar{S} \cap B\}$. Let L be a Lie algebra. A subset $K \subseteq L$ is called a ***Lie wedge*** if there exists a local semigroup S with respect to some open ε-ball B such that $K = \mathbb{L}(S)$ and a Lie wedge K is called a ***Lie semialgebra*** if $K \cap B$ is a local semigroup with respect to some open ε-ball B.

Remark. Let S be a locally compact C_s^∞ monoid. Then there is a local Lie group embedding of S (Theorem 6.1 from [1]). Let $f : U \subset S \to G$ be a local Lie group embedding of S, let $L(f) : L(S) \to L(G)$ be an isomorphism of Lie algebras induced by f, and let B be a small open neighborhood in $L(G)$ such that the multiplication * is defined on $B \times B$ and exponential mapping restricted to B is a diffeomorphism into G. Then $f \circ \exp_S = \exp_G \circ L(f)$ on $L(f)^{-1}(B)$ in $W(S)$.

For submonoid T of S, define $W(T) = \{X \in W(S)|\exp(tX) \in T$ for all $t \geq 0\}$.

LEMMA 2.1. *Let S be a locally compact C_s^∞ monoid. Then* $\exp : W(S) \to S$ *is a continuous map.*

Proof. Let U be an open set containing $\exp(X)$ and let B be an open set in $W(S)$ containing 0 which is diffeomorphism onto B_s in S

containing identity. Then there exists positive integer n such that $\frac{1}{n}X \in B$ and $\exp(\frac{1}{n}X) \in B_s$. And since $\exp(\frac{1}{n}X)^n = \exp(X) \in U$, there exists open subset V of S such that $\exp(\frac{1}{n}X) \in V$ and $V^n \subseteq U$. Let $P = B_s \cap V$. Then P is open in B_s containing $\exp(\frac{1}{n}X)$ and so there exists open set Q in $W(S)$ such that $\frac{1}{n}X \in Q$, Q is homeomorphic to P. Thus $X \in nQ$ and $\exp(nQ) = \exp(Q)^n = P^n \subseteq V^n \subseteq U$.

THEOREM 2. *Let S be a locally compact C_s^∞ monoid with its Lie algebra $L(S)$. If $W(S)$ is a Lie semialgebra, then $W(T)$ is a Lie wedge for every closed submonoid T of S.*

Proof. Since $W(S)$ is Lie semialgebra, $W(S) \cap B'$ is a local semigroup for some open ε'-ball B'. Let B_δ be an open δ-ball around 0 contained in $L(f)^{-1}(B) \cap B'$ and let $K = \exp_S^{-1}(T) \cap B_\delta$. Then if $x, y \in K$ and $x * y \in L(f)^{-1}(B) \cap B_\delta$, then $x, y \in W(S) \cap B', x * y \in B'$ and so $x * y \in W(S)$ since $W(S) \cap B'$ is local semigroup. Thus $x * y \in L(f)^{-1}(B) \cap W(S)$. Note that $\exp_S(x * y) = (f^{-1} \circ \exp_G \circ L(f))(x * y) = (f^{-1} \circ \exp_G)(L(f)(x) * L(f)(y)) = f^{-1}(\exp_G(L(f)(x)) \exp_G(L(f)(y))) = (f^{-1} \circ \exp_G \circ L(f))(x)(f^{-1} \circ \exp_G \circ L(f))(y) = \exp_S(x) \exp_S(y) \in T$. Hence $x * y \in K$ and so K is local semigroup with respect to B_δ and closed in B_δ since exponential map is continuous (Lemma 2.1.). Now $X \in \mathbb{L}(K)$ if and only if $\mathbb{R}^+ X \cap B_\delta \subseteq \bar{K} \cap B_\delta = K$ if and only if $\exp(tX) \in T$ for small positive t if and only if $\exp(tX) \in T$ for all non-negative t if and only if $X \in W(T)$. Thus we have $W(T) = \mathbb{L}(K)$

COROLLARY 2.1. *Let S be a locally compact C_s^∞ monoid with its Lie algebra $L(S)$. If $W(S)$ is a Lie semialgebra, then there is an open neighborhood Q around 0 in $L(S)$ such that $*$ is defined on $Q \times Q$ and that*

(1) $\exp|_{W(S) \cap Q}$ *is a C_s^∞ diffeomorphism into S.*
(2) $\exp(x * y) = \exp(x) \exp(y)$ *for $x, y \in W(S) \cap Q$ and $x * y \in Q$.*

Proof. In the proof of Theorem 2., let $Q = B_\delta$.

COROLLARY 2.2. *If S is a locally compact commutative C_s^∞ monoid, then $W(T)$ is a Lie wedge for every closed submonoid T of S.*

COROLLARY 2.3. *If S is a locally compact 2-dimensional C_s^∞ monoid, then $W(T)$ is a Lie wedge for every closed submonoid T of S.*

Proof. Note that any wedge in 2-dimensional Lie algebra is a Lie semialgebra.

For a monoid S, let $H(1)$ be the group of units of S. Then $H(1)$ is expressed by $\{x \in S | 1 \in xS \cap Sx\}$.

LEMMA 3.1. ([1, II. Proposition 2.2.]) *Let G be a C_s^k group and let $\theta : G \to G$ be the inversion map. Then θ is a C_s^k map.*

THEOREM 3. *Let S be a C_s^∞ monoid with its Lie algebra $L(S)$. If $H(1)$ is a C_s^1 group, then $W(H(1)) = W(S) \cap -W(S)$.*

Proof. Since $H(1) \subseteq S$, $W(H(1)) \subseteq W(S)$. Suppose that $X \in W(H(1))$. Then $\exp(tX) \in H(1)$ for all $t \in \mathbb{R}^+$. Let $\alpha(t) = \exp(tX)$ for all $t \in \mathbb{R}^+$ and let $\theta : H(1) \to H(1)$ be the inversion map. Define a map $\beta : \mathbb{R}^+ \to H(1)$ by $\beta(t) = \theta \circ \alpha(t)$. Then β is well-defined, and continuous homomorphism and so C_s^∞ map ([5, Theorem 2.]). Thus there exists $Y \in W(S)$ such that $\beta(t) = \exp(tY)$ for all $t \in \mathbb{R}^+$. Now $Y_1 = \beta'(0) = d\beta(0)(1) = d(\theta \circ \alpha(t))(0)(1) = d\theta(1)(d\alpha(0)(1)) = -X_1$. Thus $Y = -X$ and so $-X \in W(S)$ and we have $W(H(1)) \subseteq W(S) \cap -W(S)$. If $X \in W(S) \cap -W(S)$, then $tX \in W(S)$ for all $t \in \mathbb{R}$. Thus $\exp(tX) \in S$ for all $t \in \mathbb{R}$. So $1 = \exp(tX)\exp(-tX) = \exp(-tX)\exp(tX)$ for all $t \geq 0$. Hence we have $\exp(tX) \in H(1)$ for all $t \geq 0$ and so $X \in W(H(1))$. Thus $W(S) \cap -W(S) \subseteq W(H(1))$.

COROLLARY 3.1. *Let S be a locally compact commutative C_s^∞ monoid (or locally compact 2-dimensional C_s^∞ monoid) with its Lie algebra $L(S)$. If $H(1)$ is a C_s^1 group, then $W(H(1))$ is a subalgebra of $L(S)$.*

It is well known that if S is a divisible and compact monoid, then $H(1)$ is connected. The following theorem shows that necessary and sufficient condition for $H(1)$ to be connected in a compact C_s^∞ monoid is $H(1)$ is contained in the subsemigroup of S which is generated by $\exp W(S)$.

THEOREM 4. *Let S be a compact C_s^∞ monoid. Then $H(1) \subseteq \langle \exp W(S) \rangle$ if and only if $H(1)$ is connected.*

Proof. Let $g \in H(1)$. Then $g = \exp(X_1)\exp(X_2)\cdots\exp(X_n)$ for some $X_1, X_2, \ldots, X_n \in W(S)$ and $\exp(X_i) \in H(1)$ since S is compact monoid for $i = 1, 2, \cdots, n$. Now for $i = 1, 2, \cdots, n$, $\exp((1-t)X_i)\exp(tX_i) = \exp(X_i) \in H(1)$ for all $t(0 \le t \le 1)$. Thus $\exp(tX_i) \in H(1)$ for all $t(0 \le t \le 1)$. So we have for each $i = 1, 2, \cdots, n, \exp(tX_i) \in H(1)$ for all $t \in \mathbb{R}^+$. Hence $X_1, X_2, \cdots, X_n \in W(H(1)), H(1) =< \exp W(H(1)) >$ and so $H(1)$ is connected. Conversely, $H(1)$ is a compact connected topological group with no small subgroup and therefore is connected Lie group.

In [1], George E. Graham suggested an open problem relating to homomorphism of C_s^k semigroup.

PROBLEM. Let S and T be C_s^∞ monoids and let $f : S \to T$ be a continuous homomorphism. Must f be a C_s^∞ differentiable on a neighborhood of 1 (identity of S)?

The following theorem is a partial solution for the problem.

If S is a C_s^k semigroup, then the tangent space $T_{(s_1, s_2, \cdots, s_n)} S \times S \times \cdots \times S$ may be identified with $T_{s_1}S \times T_{s_2}S \times \cdots \times T_{s_n}S$

LEMMA 5.1. *Let S be a C_s^k monoid. Define a map $m_n : S \times S \times \cdots \times S \to S$ by $m_n(s_1, s_2, \ldots, s_n) = s_1 s_2 \cdots s_n$, then $dm_n(1, 1, \ldots, 1)(v_1, v_2, \ldots, v_n) = v_1 + v_2 + \cdots + v_n$ for all $(v_1, v_2, \cdots, v_n) \in T_1S \times T_1S \times \cdots \times T_1S$.*

Proof. For $n = 2$, $dm(1,1)(v_1, v_2) = d\rho_1(1)(v_1) + d\lambda_1(1)(v_2) = v_1 + v_2$ by product rule [1]. Suppose that the assertion is true for $n-1$. Note that $m_n = m \circ (1_s \times m_{n-1})$, where m is multiplication on S and $m_{n-1} : S \times \cdots \times S \to S$ by $m_{n-1}(s_2, \ldots, s_n) = s_2 \ldots s_n$. Hence

$$\begin{aligned} dm_n(1, 1, \ldots, 1) &= dm(1,1) \circ d(1_s \times m_{n-1})(1, 1, \ldots, 1) \\ &= dm(1,1) \circ (d1_s \times dm_{n-1})(1, 1, \ldots, 1). \end{aligned}$$

Thus we have

$$\begin{aligned} &dm_n(1, 1, \ldots, 1)(v_1, v_2, \ldots, v_n) \\ &\qquad = dm(1,1)(v_1, dm_{n-1}(1, 1, \ldots, 1)(v_2, \ldots, v_n)) \\ &\qquad = v_1 + dm_{n-1}(1, 1, \ldots, 1)(v_2, \ldots, v_n) \\ &\qquad = v_1 + v_2 + \cdots + v_n. \end{aligned}$$

for all $(v_1, v_2, \ldots, v_n) \in T_1S \times \cdots \times T_1S$.

LEMMA 5.2. (Inverse Function Theorem [1, Theorem 4.1]). *Let M and N be C_s^k manifolds, let $f : M \to N$ be a C_s^k map and let $p \in M$. If $df(p)$ is an isomorphism onto $T_{f(p)}N$, then there is an open set U about p such that $f|_U$ is a diffeomorphism onto the admissible subset $f(U)$ of N, $(f|_U)^{-1}$ is a C_s^k map.*

THEOREM 5. *Let S be a locally compact C_s^∞ monoid with $L(S) = < W(S) >$ as a vector space and T be a C_s^∞ monoid. Then every continuous homomorphism $f : S \to T$ is a C_s^∞ map on a neighborhood (admissible subset of S) of the identity of S.*

Proof. Let $X_1, X_2, \ldots, X_n \in W(S)$ such that $< X_1, X_2, \ldots, X_n > = L(S)$ as a vector space. For each i, $\alpha_i : \mathbb{R}^+ \to T$ by $\alpha_i(t) = f(\exp(tX_i))$ is continuous homomorphism since exp map is continuous and so they are C_s^∞ map [5,Theorem 2]. Thus there exist elements $Y_1, Y_2, \ldots, Y_n \in W(T)$ such that $\exp(tY_i) = \alpha_i(t)$ for $i = 1, 2, \ldots, n$. Define a map $F : (\mathbb{R}^+)^n \to S$ by $F(t_1, \ldots, t_n) = \exp(t_1X_1)\ldots\exp(t_nX_n)$, then since $f \circ F(t_1, \ldots, t_n) = \alpha_1(t_1)\ldots\alpha_n(t_n)$, $f \circ F$ is a C_s^∞ map. Now, $F = m \circ (\beta_1 \times \beta_2 \times \cdots \times \beta_n)$ where $\beta_i : \mathbb{R}^+ \to S$ by $\beta_i(t) = \exp(tX_i)$ for $i = 1, 2, \ldots, n$. Let e_j denote the n-tuple with 1 in the j-th place and 0's otherwise. Then

$$\begin{aligned} dF(0)(e_j) &= d(m \circ \beta_1 \times \cdots \times \beta_n)(0)(e_j) \\ &= dm(1, 1, \ldots, 1)((d\beta_1(0) \times \cdots \times d\beta_n(0))(e_j)) \\ &= dm(1, 1, \ldots, 1)(0, 0, \ldots, 0, d\beta_j(0)(1), 0, \ldots, 0) \\ &= d\beta_j(0)(1) = \beta_j'(0) = X_j(1). \end{aligned}$$

Hence $dF(0)$ maps an n-dimensional basis of $\mathbb{R}^n$ into the n-dimensional basis of $T_1S \simeq L(S)$ and so it is linear isomorphism. Thus by Lemma 5.2., there exists an open set U about 0 in $(\mathbb{R}^+)^n$ such that $F|_U$ is a diffeomorphism onto the admissible subset $F(U)$ of S, $F|_U^{-1}$ is a C_s^∞ map. Thus $f = (f \circ F) \circ F|_U^{-1}$ on $F(U)$ and so f is a C_s^∞ map in a neighborhood (admissible subset) of the identity of S.

Remark. Let S and T be C_s^k monoids and let $f : S \to T$ be a C_s^k local homomorphism. Define $L(f) : L(S) \to L(T)$ by $L(f)(X)_p = d\rho_p(1)(df(1)(X_1))$. Then $L(f)$ is a Lie algebra homomorphism if $k > 2$.

COROLLARY 5.1. *Let S be a locally compact C_s^∞ monoid with $L(S) =< W(S) >$ as a vector space and T be a C_s^∞ monoid. Suppose that $f : S \to T$ be a continuous homomorphism. Then f induces Lie algebra homomorphism $L(f)$ and if $A = \operatorname{Ker} f$, then $W(A) \cap -W(A)$ is a Lie subalgebra of $L(S)$.*

Proof. Note that A is a closed subset of S. If $X \in W(A) \cap -W(A)$, then $\exp(tX) \in A$ for all $t \in \mathbb{R}$ if and only if $f(\exp(tX)) = 1_T$ for all $t \in \mathbb{R}$. Let $\alpha(t) = f \circ \beta(t)$ where $\beta(t) = \exp(tX)$ for all $t \in \mathbb{R}$. Then α is a one-parameter submonoid of T and $\alpha'(0) = d\alpha(0)(1) = df(1)(d\beta(t)(0)(1)) = df(1)(\beta'(0)(1)) = df(1)(X_1) = d\rho_1(1)(df(1)(X_1)) = L(f)(X)_1$. It follows that $L(f)(X) \in W(T)$ and $\alpha(t) = \exp(L(f)(tX))$ for all $t \in \mathbb{R}$. So we have $f(\exp(tX)) = 1_T$ for all $t \in \mathbb{R}$ if and only if $\exp(tL(f)(X)) = 1_T$ for all $t \in \mathbb{R}$ if and only if $L(f)(X) = 0$. Thus $W(A) \cap -W(A) = \operatorname{Ker} L(f)$.

A *ray semigroup* is a C_s^∞ monoid S such that S is generated by the set of all elements of S of the form $\alpha(t)$, where α is a C_s^∞ one-parameter submonoid of S and $t \geq 0$.

COROLLARY 5.2. *Let S be a locally compact commutative ray semigroup and let T be a C_s^∞ monoid. Then every continuous homomorphism $f : S \to T$ is a C_s^∞ map on a neighborhood (admissible subset) of the identity of S.*

Proof. Since S is a finite dimensional ray semigroup, $W(S)$ generates $L(S)$ as a Lie algebra ([1], II. Corollary 6.3.). And since S is a commutative ($L(S)$ is commutative), $W(S) - W(S) = L(S)$.

COROLLARY 5.3. *Let S be a 2-dimensional ray semigroup and let T be a C_s^∞ monoid. Then every continuous homomorphism $f : S \to T$ is a local C_s^∞ map on a neighborhood (admissible subset) of the identity of S.*

Proof. Since S is a ray semigroup, $S =< \exp W(S) >$. If the dimension of $W(S)$ is 1, then for all $Y \in W(S)$, $Y = tX$ for some $t \in \mathbb{R}^+$ and fixed $X \in W(S)$. Thus $S =< \exp W(S) >= \{\exp(tX) | t \geq 0\}$ and contradict to 2-dimensional ray semigroup. Hence $L(S) =< W(S) >$ as a vector space.

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