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S^1-RATIONAL HOMOTOPY THEORY AND ITS APPLICATIONS

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Introduction

D. Sullivan's rational homotopy theory establishes an equivalence between the rational homotopy category of (1-connected) topological spaces to the homotopy category of (1-connected) commutative $\mathbb{Z}_+$-graded differential algebras over $\mathbb{Q}$ which are minimal by providing a functor from one category to another.

The purpose of this article is to develop rational homotopy theory of S^1-spaces (spaces equipped with S^1 actions) on the line of Sullivan's rational homotopy theory.

Let k be a field of characteristic 0. Let X be a graded space. ΛX will denote the free graded commutative algebra over X :

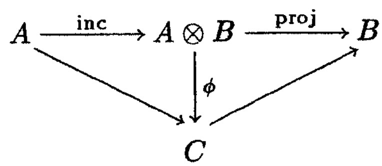
$$\Lambda X = \text{exterior algebra}(X^{\text{odd}}) \otimes \text{symmetric algebra}(X^{\text{even}}).$$

A Koszul-Sullivan(KS) extension is a sequence of differential graded algebras(DGA's)

$$(A, d_A) \to (C, d_C) \to (B, d_B)$$

such that

1) $B = \Lambda X$ for some graded space X
2) there is a commutative diagram of graded algebra homomorphisms



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where ϕ is an isomorphism of graded algebras.

3) there is a well ordered homogeneous basis $\{x_\alpha\}_{\alpha \in I}$ for X with

$$d_C x_\alpha \in A \otimes B_{<\alpha}$$

where $B_{<\alpha}$ is the subalgebra generated by $(x_\beta)_{\beta<\alpha}$.

A KS-extension is called minimal if the basis can be chosen so that $\deg x_\beta < \deg x_\alpha$ implies $\beta < \alpha$. If A is k (so that $C = B = \Lambda X$) C is called a KS-complex (resp. a minimal KS-complex). Given a DGA map $f : (A, d_A) \to (A', d'_A)$ a KS-model for f is a commutative diagram of DGA's

$$\begin{array}{ccccc} & \overset{f}{\nearrow} & (A', d'_A) & & \\ & & \uparrow{\scriptstyle \phi} & & \\ (A, d_A) & \xrightarrow{\ i\ } & (C, d_C) & \xrightarrow{\ p\ } & (B, d_B) \end{array}$$

where (i, p) is a KS-extension and $\phi^* : H(C) \to H(A')$ is an isomorphism. If the extension is minimal, it is called a minimal model for f. If $A = k$ (so that f is the inclusion of k in A') (C, ϕ) or simply C is called a model of A'.

Let M be a topological space and recall that Sing M denotes the simplicial set of singular simplices on M. Then we can form the DGA $(A(\operatorname{Sing} M), d)$ which will be denoted simply by $(A(M), d)$ and called the DGA of differential forms on M. A minimal model of $A(M)$ is called a minimal model of M.

If $p : E \to B$ is a fibration and (A, d) is a minimal model of B then a model for E is given [3] by a KS-extension $(A \otimes \Lambda X, D)$ for some graded vector space X where $D|A = d$.

If S^1 acts on a space X the action is studied via the fibration $X \to ES^1 \times_{S^1} X \to BS^1$ introduced by Borel. It leads [5] to a sequence

$$(*) \qquad (k[u], 0) \to (k[u] \otimes \Lambda V, D) \to (\Lambda V, d)$$

of morphisms of KS-complexes in which KS-complexes are respectively a minimal model for BS^1, a model for $ES^1 \times_{S^1} X$ and a minimal model for X. $(*)$ is called a KS-extension associated to the S^1-action. Note that if $V^1 = 0$ then the model for $ES^1 \times_{S^1} X$ is also minimal. In fact the differential D on $k[u] \otimes \Lambda V$ can be written as

$$D = d + u\beta_1 + \cdots + u^n \beta_n + \cdots$$

where β_i is a homomorphism of degree $-(2i-1)$.

In section 1 one introduces some invariants of an S^1-space using $(*)$-the periodic equivariant cohomology and the type of an action, and some results about them are given.

In section 2 one discusses the relationship between the equivariant cohomology in dimension k of an S^1-manifold $\tilde{M}$ (compact oriented n-dimensional) to the dual of equivariant cohomology in dimension $n-k-1$. This is established by a linear map which we call (equivariant) Poincaré duality. The main result shows that the failure of Poincaré duality to be an isomorphism is measured by the periodic equivariant cohomology of M.

In section 3 one reviews a model of the space of all cross sections of a (nilpotent) fibration proposed by Sullivan [7]. One also introduces the concept of homotopy fixed point set and finds a model for the set using the Sullivan model.

In section 4 one considers a model for the homotopy fixed point set of an infinite symmetric product of an S^1-space and studies a relationship with the minus equivariant cohomology defined in section 2.

1. Some invariants of an S^1-space

Let $\tilde{X}$ be an S^1-space and let $k[u] \to k[u] \otimes \Lambda V \to \Lambda V$ be a KS-extension associated with the action as in the introduction. The equivariant cohomology $H^*_{S^1}(X;k) = H^*(ES^1 \times_{S^1} X;k)$ of X can be calculated as the homology of $(k[u] \otimes \Lambda V, D)$. Note that $H^*(BS^1;k) = H^*_{S^1}(\text{point}\,;k) = k[u]$, $\deg u = 2$.

DEFINITION 1.1. The periodic equivariant cohomology of X is defined as the direct limit

$$PH_{S^1}{}^*(X) = \varinjlim(\cdots \to H_{S^1}{}^*(X) \xrightarrow{\cdot u} H_{S^1}{}^{*+2}(X) \to \cdots).$$

And an S^1-map $f : \tilde{X} \to \tilde{Y}$ of S^1-spaces $\tilde{X}$ and $\tilde{Y}$ is called periodic equivariant cohomology equivalence (PEH-equivalence) if the induced map

$$Pf : PH_{S^1}{}^*(Y) \to PH_{S^1}{}^*(X)$$

is an isomorphism for each $*$.

REMARK 1.2. Clearly we have $PH_{S^1}{}^*(X) = PH_{S^1}{}^{*+2}(X)$ i.e., $PH_{S^1}{}^*(X)$ is a $\mathbb{Z}_2$-graded vector space. And $f : \tilde{X} \to \tilde{Y}$ is a PEH-equivalence if and only if the kernel and cokernel of $f^* : H_{S^1}{}^*(Y) \to H_{S^1}{}^*(X)$ are torsion $k[u]$-modules.

THEOREM 1.3. *If the action on a finite dimensional S^1-space $\tilde{X}$ is almost free, then $PH_{S^1}{}^*(X) = 0$ for all $*$.*

Proof. The hypothesis implies that the map $ES^1 \times_{S^1} X \to X/S^1$ induced by the projection $ES^1 \times_{S^1} X \to X$ is a rational homotopy equivalence. Hence $H^n_{S^1}(X) = H^n(X/S^1) = 0$ for sufficiently large n since X is finite dimensional. This completes the proof.

Let x_0 be the base point of a 2-connected topological space X and let

$$\Omega^2(X) = \{f : S^2 \to X \mid f \text{ is continuous and } f(\text{south pole}) = x_0\}$$

be the double loop space. We have a natural S^1-action on $\Omega^2 X$ induced by the rotation of S^2 around the north-south axis. We also have an evaluation map $\alpha : \Omega^2 X \to \overline{X}$ defined by $\alpha(f) = f(\text{north pole})$. Clearly α is equivariant where $\overline{X}$ is equipped with the trivial action. Hence we have the following diagram

$$\begin{array}{ccccc} \Omega^2 X & \to & ES^1 \times_{S^1} \Omega^2 X & \to & BS^1 \\ \downarrow & & \downarrow & & \| \\ X & \to & BS^1 \times X & \to & BS^1 \end{array}$$

If $(\Lambda(x_\alpha), d)$ is a minimal model of X, then $(\Lambda(\overline{\overline{x}}_\alpha), 0)$ is a minimal model of $\Omega^2 X$ where $\deg \overline{\overline{x}}_\alpha = \deg x_\alpha - 2$. Thus we obtain the following commutative diagram

$$\begin{array}{ccccc} k[u] & \to & (k[u] \otimes \Lambda(\overline{\overline{x}}_\alpha), D) & \to & (\Lambda(\overline{\overline{x}}_\alpha), 0) \\ \| & & \uparrow & & \uparrow \\ k[u] & \to & (k[u] \otimes \Lambda(x_\alpha), \tilde{d}) & \to & (\Lambda(x_\alpha), d) \end{array}$$

If $\tilde{d}x_\alpha = \Sigma a_{ij}x_ix_j + \Sigma b_{lmn}x_lx_mx_n + \cdots$, then the differential D is given by

$$D\bar{\bar{x}}_\alpha = \Sigma a_{ij}\bar{\bar{x}}_i\,\bar{\bar{x}}_j\,u + \Sigma b_{lmn}\bar{\bar{x}}_l\,\bar{\bar{x}}_m\,\bar{\bar{x}}_n u^2 + \cdots .$$

Now it is easy to see that the evaluation map α is a PEH-equivalence.

DEFINITION 1.4. An S^1-space $\tilde{X}$ is called of type r if r is the smallest integer with the following property: there exists a minimal model of $ES^1 \times_{S^1} X$ of the form $(k[u] \otimes \Lambda V, D)$ where $(\Lambda V, d)$ is a minimal model of X and D is given by

$$Df = df + u\beta_1 f + \cdots + u^r\beta_r f, \quad \beta_r \neq 0$$

for all $f \in k[u] \otimes \Lambda V$.

The following lemma is well known.

LEMMA 1.5. *If S^1 acts on a finite dimensional space X, the fixed point set is non-empty if and only if*

$$H^*(BS^1;\mathbb{Q}) \to H^*(ES^1 \times_{S^1} X;\mathbb{Q})$$

is injective.

THEOREM 1.6. *Let $\tilde{X}$ be an n-connected finite dimensional S^1-space. If the action is of type $\leq (n+1)/2$, then it has a fixed point.*

Proof. Suppose that the fixed point set is empty and let m be the type of the S^1-space $\tilde{X}$. Then there exists a minimal model $(k[u] \otimes \Lambda V, D)$ of $ES^1 \times_{S^1} X$ where D is given by

$$Df = df + u\beta_1 f + \cdots + u^m\beta_m f.$$

By the lemma 1.5 there exists $\alpha = \alpha_0 + \alpha_1 u + \cdots + \alpha_p u^p$ in $k[u] \otimes \Lambda V$ and an integer r such that

$$\begin{aligned} u^r &= D(\alpha_0 + \alpha_1 u + \cdots + \alpha_p u^p),\ \deg \alpha_i > n,\ i = 0, 1, \ldots, p \\ &= (d\alpha_0 + u\beta_1\alpha_0 + \cdots + u^m\beta_m\alpha_0) \\ &\quad + (d\alpha_1 + u\beta_1\alpha_1 + \cdots + u^m\beta_m\alpha_1) \cdot u \\ &\quad \cdots \\ &\quad + (d\alpha_p + u\beta_1\alpha_p + \cdots + u^m\beta_m\alpha_p) \cdot u^p \end{aligned}$$

If $r \leq m$, then we have $\beta_r\alpha_0 + \beta_{r-1}\alpha_1 + \cdots = 1$ where β_k decreases the degree by $2k-1$ and $\deg \alpha_k = 2r-2k-1$. Without loss of generality we may assume that $\beta_r\alpha_0 \neq 0$. (If so, we take $\alpha = \alpha_1 + \alpha_2 u + \cdots + \alpha_{p-1}u^{p-1}$ to have $u^{r-1} = D\alpha$.) Note that $\deg \alpha_0 = 2r-1 > n$ since X is n-connected. Hence $2m-1 \geq 2r-1 > n$ which implies that $m > (n+1)/2$. If $r > m$, then $\beta_m\alpha_{r-m} + \beta_{m-1}\alpha_{r-m+1} + \cdots = 1$, which implies that $\deg \alpha_{r-m} = 2m-1 > n$. (If $\beta_m\alpha_{r-m} = 0$, we take $\alpha = \alpha_{r-m+1}u^{r-m+1} + \cdots + \alpha_p u^p$ and $u^m = D\alpha$.) This proves the theorem.

2. Poincaré Duality

Let $\tilde{M}^n$ be a compact oriented n-dimensional S^1-manifold and let $(\Omega^*(M), d)$ be the De Rham complex. Let X be the vector field generated by the S^1-action. We denote $i_X : \Omega^*(M) \to \Omega^{*-1}(M)$ to be the contraction along the vector field X defined by

$$i_X w(X_1, \ldots, X_{p-1}) = w(X, X_1, \ldots, X_{p-1})$$

for $w \in \Omega^p(M)$. Then clearly $i_X{}^2 = 0$. We define $L_X = di_X + i_X d : \Omega^*(M) \to \Omega^*(M)$, the Lie derivative with respect to X.

DEFINITION 2.1. $\Omega_{S^1}{}^*(M) = \{w \in \Omega^*(M) \,|\, L_X w = 0\}$. Such forms are called invariant forms.

In fact, by choosing a model $\Omega_{S^1}{}^*(M)$ for M instead of a minimal one we can take [2] an extension

$$\mathbb{R}[u] \to (\mathbb{R}[u] \otimes \Omega_{S^1}{}^*(M), D) \to (\Omega_{S^1}{}^*(M), d)$$

as a model of the Borel fibration where D has the form $D = d + i_X u$. This gives us the following.

LEMMA 2.2. (1) $H(\oplus_{n\geq 0}\Omega_{S^1}{}^{*-2n}(M), d + i_X) = H_{S^1}(M; \mathbb{R})$.
(2) $PH_{S^1}{}^*(M; \mathbb{R}) = H(\oplus_n \Omega_{S^1}{}^{*+2n}(M), d + i_X)$.

Let $\tilde{M}^n$ be as above equipped with an invariant metric. Then $*$ operator

$$* : \Omega_{S^1}{}^k(M) \to \Omega_{S^1}{}^{n-k}(M)$$

induced by the metric sends the invariant forms to the invariant forms and satisfies $*^2 = id$. We have the following commutative diagrams

$$
\begin{array}{ccccccccc}
\longrightarrow & \Omega_{S^1}{}^{r}(M) & \xrightarrow{d} & \Omega_{S^1}{}^{r+1}(M) & \xrightarrow{d} & \Omega_{S^1}{}^{r+2}(M) & \longrightarrow \\
 & *\downarrow & & *\downarrow & & *\downarrow & \\
\longrightarrow & \Omega_{S^1}{}^{n-r}(M) & \xrightarrow{\delta} & \Omega_{S^1}{}^{n-r-1}(M) & \xrightarrow{\delta} & \Omega_{S^1}{}^{n-r-2}(M) & \longrightarrow
\end{array}
$$

where $\delta = *d*$ and $\beta = *i_X*$. Note that $\delta^2 = \beta^2 = 0$ and $\delta\beta + \beta\delta = 0$.

Then we have the following commutative diagram.

$$
\begin{array}{cccc}
\oplus_{i\geq 0}\Omega_{S^1}{}^{n-k+2i}(M) & \xrightarrow{d+i_X} & \oplus_{i\geq 0}\Omega_{S^1}{}^{n-k+1+2i}(M) & (1) \\
*\downarrow & & *\downarrow & \\
\oplus_{i\geq 0}\Omega_{S^1}{}^{k-2i}(M) & \xrightarrow{\delta+\beta} & \oplus_{i\geq 0}\Omega_{S^1}{}^{k-1-2i}(M) & (2)
\end{array}
$$

DEFINITION 2.4. The above complex (1) ((2)) defines so called the minus equivariant cohomology (equivariant homology) of M denoted by $-\mathbb{H}_{S^1}{}^{n-k}(M;\mathbb{R})$ $(H_k{}^{S^1}(M;\mathbb{R}))$ respectively. In particular $*$ operator induces an isomorphism on cohomology

$$-\mathbb{H}_{S^1}{}^{n-k}(M;\mathbb{R}) \cong H_k{}^{S^1}(M;\mathbb{R}).$$

Now we have the following commutative diagram of short exact sequence;

$$
\begin{array}{ccccccccc}
0 & \to & \oplus_{i>0}\Omega_{S^1}{}^{k-2i}(M) & \to & \oplus_{i}\Omega_{S^1}{}^{k+2i}(M) & \to & \oplus_{i\geq 0}\Omega_{S^1}{}^{k+2i}(M) & \to & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \to & \oplus_{i>0}\Omega_{S^1}{}^{k-2i}(M) & \to & \oplus_{i\geq 0}\Omega_{S^1}{}^{k-2i}(M) & \to & \Omega_{S^1}{}^{k}(M) & \to & 0
\end{array}
$$

This induces a commutative diagram of a long exact sequence in cohomology.

$$
\begin{array}{ccccccccc}
\to PH_{S^1}{}^{k}(M) & \longrightarrow & -\mathbb{H}_{S^1}{}^{k}(M) & & \longrightarrow & & H_{S^1}{}^{k-1}(M) & \longrightarrow & PH_{S^1}{}^{k+1}(M) \to \\
\uparrow & & \uparrow & \searrow^{\cong} & & \nearrow_{\widetilde{D}} & \| & & \uparrow \\
 & & & & H_{n-k}{}^{S^1}(M) & & & & \\
\longrightarrow H_{S^1}{}^{k}(M) & \longrightarrow & H^k(M) & & \longrightarrow & & H_{S^1}{}^{k-1}(M) & \longrightarrow & H_{S^1}{}^{k+1}(M) \longrightarrow
\end{array}
$$

The homomorphism $\tilde{D}$ is called the equivariant Poincaré duality map and the failure of $\tilde{D}$ to be an isomorphism is measured by $PH_{S^1}{}^*(M)$. Note that the bottom level is Gysin sequence. Note also that if the action on M is almost free then $H_*{}^{S^1}(M) \cong H_*(M/S^1)$ and $H_{S^1}{}^*(M) \cong H^*(M/S^1)$. Since $PH_{S^1}{}^*(M) = 0$ if the action is free we finally note the above map $\tilde{D}$ is in fact the Poincaré-duality isomorphism for the orbit space M/S^1.

3. Sullivan model for a homotopy fixed point set

Let $p : E \to B$ be a fibration and let $A \to \Omega^*(B)$ be a minimal model for B. As we have seen in the introduction a model for E is given by $(A \otimes \Lambda V, D)$ for some graded vector space V where $D|A = d$, the differential on A. Suppose $s : B \to E$ be a section. Sullivan [7] proposed an algebraic model for the space Γ_s of section in the connected component of s as following:

Let W be the set all pairs (a, b^*) where a is a generator of V and b^* is a (dual) additive generator of A. The degree of (a, b^*) is $\deg a - \deg b^*$. We set (a, b^*) of negative degree equal to 0 and convert (a, b^*) of degree 0 into scalars by $(a, b^*) = \langle s(a), b^* \rangle$. Let $\Gamma_0 = \Lambda W$. We have a universal evaluation map $e : B \times \Gamma_s \to E$ defined by $e(b, \sigma) = \sigma(b)$, $\sigma \in \Gamma_s$, $b \in B$. This gives a DGA map $e^* : A \otimes \Lambda V \to A \otimes \Gamma_0$ defined by $e^*(a) = \Sigma_b (a, b^*) b$ which forces the definition of $\mathbb{D}(a, b^*) = (ad, b^*) \pm (a, \partial b^*)$. ($\partial$ is the dual of d.)

A. Haefliger [4] showed that the above differential graded algebra is indeed a model for the space of sections.

DEFINITION 3.1. Let $\tilde{X}$ be an S^1-space. The homotopy fixed point set X^{hS^1} is defined to be the space of sections of the associated Borel fibration $X \to ES^1 \times_{S^1} X \xrightarrow{\pi} BS^1$.

REMARK 3.2. $X^{hS^1} = \mathrm{Maps}_{S^1}(ES^1, X)$, S^1-maps from ES^1 to X and there is an inclusion $i : X^{S^1} \to X^{hS^1}$ of the fixed point set into X^{hS^1}.

THEOREM 3.3. *Let $\tilde{X}$ be finite dimensional S^1-space. Then $X^{S^1} \neq \phi$ if and only if $X^{hS^1} \neq \phi$.*

Proof. If $X^{S^1} \neq \phi$, say $x \in X^{S^1}$, then one can construct a map $BS^1 \to ES^1 \times_{S^1} X$ sending $[e]$ to $[e, x]$. Conversely if $X^{hS^1} \neq \phi$, then

$\pi^* : H^*(BS^1) \to H^*(ES^1 \times_{S^1} X)$ is injective. We have already seen that this is equivalent to $X^{S^1} \neq \phi$, and the proof is complete.

Now we can apply Sullivan's construction to obtain a model for the homotopy fixed point set of an S^1-space. If

$$k[u] \to (k[u] \otimes \Lambda V, D) \to (\Lambda V, d)$$

is a KS-extension associated with the action, the generators are of the form (x_α, u^n) where x_α is a generator of V and the differential $\mathbb{D}$ is simply defined by

$$(**) \qquad \mathbb{D}(x, u) = (Dx, u^n)$$

REMARK 3.4. We need a following convention for $(**)$ to make sense:

$$(x_1 x_2 \cdots x_r, u^n) = \sum_{\substack{i_1 + \cdots + i_r = n \\ \deg x_j - 2i_j \geq 0,\ \forall j}} (x_1, u^{i_1})(x_2, u^{i_2}) \cdots (x_r, u^{i_r}).$$

4. A model for $P^\infty(\tilde{M})^{hS^1}$

Let $\tilde{M}$ be a free S^1-manifold and let $P^\infty(M)$ be the infinite symmetric product of M, which has the induced S^1-action. By P. May [6] we have an isomorphism

$$(***) \qquad \pi_*(P^\infty(M)) \cong H_*(M)$$

We also have the following diagram

$$\begin{array}{ccccc} M & \to & ES^1 \times_{S^1} M & \to & BS^1 \\ \uparrow i & & \uparrow \tilde{i} & & \| \\ P^\infty(M) & \to & ES_1 \times_{S^1} P^\infty(M) & \to & BS^1 \end{array}$$

where i and $\tilde{i}$ are defined up to homotopy.

Let

$$W^* = \ker(\Omega_{S^1}{}^*(M) \xrightarrow{d} \Omega_{S^1}{}^{*+1}(M))$$
$$(W')^* = \mathrm{im}(\Omega_{S^1}{}^{*-1}(M) \xrightarrow{d} \Omega_{S^1}{}^*(M))$$
$$\text{and} \quad (W'')^* = \sigma_*(H^*(M))$$

for a section $\sigma_* : H^*(M) \to W^*$.

We consider a free algebra $A = (\Lambda(W'' \oplus W' \oplus \overline{W}'), \delta)$ where δ is given by $\delta w'' = 0 = \delta w'$ and $\delta \overline{w}' = w'$ for $w'' \in W''$, $w' \in W''$ and $\overline{w}' \in \overline{W}'$ and $(\overline{W}')^*$ is the graded vector space such that $(\overline{W}')^i = (W')^{i+1}$. Since $A \sim ((W''), 0)$ and $((W' \oplus \overline{W}'), \delta)$ is acyclic $(***)$ implies that A is a model for $P^\infty(M)$. Hence we obtain a model $(k[u] \otimes A, D)$ for $ES^1 \times_{S^1} P^\infty(M)$ where D can be defined as followings:

$$D = \delta + u\beta, \quad \beta w'' = i_X w'', \quad w'' \in W''$$
$$\beta w' = i_X w', \quad w' \in W' \quad \text{and} \quad \beta \overline{w}' = \overline{-i_X w'}, \quad \overline{w}' \in \overline{W}'$$

Following our previous description of Sullivan model the generators of the homotopy fixed point set $P^\infty(M)^{hS^1}$ of $P^\infty(M)$ are of the following types:

$$[w'', i], [w', i], [\overline{w}', i] \quad (i = 0, 1, 2, \dots)$$

where $w'' \in W''$, $w' \in W'$ and $\overline{w}' \in \overline{W}'$. Note that $[w'', i]$ denotes (w'', u^i) in our previous notation. The direct computation using the universal evaluation map gives us the differential $\mathbb{D}$ as following:

$$\mathbb{D}[w'', 0] = 0, \ \mathbb{D}[w'', i] = [i_X w'', i-1], \ i > 0$$
$$\mathbb{D}[w', 0] = 0, \ \mathbb{D}[w', i] = [i_X w', i-1], \ i > 0$$
$$\mathbb{D}[\overline{w}', 0] = [w', 0], \ \mathbb{D}[\overline{w}', i] = [w', i] - [\overline{i_X w'}, i-1], \ i > 0.$$

Recall that $-\mathbb{H}_{S^1}{}^*(M)$ is defined by $H(-C^*, d + i_X)$ where $-C^* = \Omega_{S^1}{}^*(M) \oplus \Omega_{S^1}{}^{*+2}(M) \oplus \cdots$. We denote U'', U' and $\overline{U}'$ the vector spaces generated by the generators $[w'', i], [w', i]$ and $[\overline{w}', i]$ respectively.

We define a chain map

$$\Phi : (U'' \oplus U' \oplus \overline{U}', \mathbb{D})^n \to (-C^n, d + i_X)$$

by

$$\Phi[w'', i] = (0, 0 \cdots 0, \overset{i+1}{w}, 0 \cdots 0)$$
$$\Phi[w', i] = (0, 0 \cdots 0, \overset{i+1}{dsw'}, 0 \cdots 0)$$
$$\Phi[\overline{w}', i] = (0, 0 \cdots 0, \overset{i+1}{sw'}, 0 \cdots 0)$$

REMARK 4.1. s is a map on $(W')^*$ to $\Omega_{S^1}{}^{*-1}(M)$ and we will not discuss about the construction of s.

THEOREM 4.2. *Φ^* is injective.*

Proof. Let $\alpha \in U'' \oplus U'' \oplus \overline{U}'$. Then α has a form

$$\alpha = \sum_i a_i[w_i'', i] + \sum_j b_j[w_j', j] + \sum_k c_k[\overline{w'_{\alpha_k}}, k],$$

and

$$Da = \sum_i a_i[i_X W_i'', i-1] + \sum_j b_j[i_X w_j', j-1] + \sum_k c_k[w'_{a_k}, k-1] - [\overline{i_x w'_{a_k}}, k-1]).$$

Hence

$$Da = 0 \Leftrightarrow \begin{bmatrix} i_X w_i'' = 0 \\ w'_{a_k} = i_X w_j', \ c_k = -b_j, \text{ some } j \end{bmatrix} \text{ for all } i \text{ and } k$$
$$\Leftrightarrow \alpha = \sum a_i[w_i'', i] + \sum_{i_x w_j = 0} b_j[w_j', j] + \sum_{i_x w_j \neq 0} b_j[w_j', j] + \sum c_k[\overline{w'_{a_k}}, k]$$
$$= \sum a_i[w_i'', i] + \sum_{i_X w_j' = 0} b_j[w_j', j] + \sum d_l([w_l', 1] - [\overline{i_X w_l'}, l-1])$$
$$= \sum a_i[w_i'', i] + \mathbb{D}\Omega \text{ for some } \Omega,$$
$$\text{and } i_X w_i'' = 0, \text{ for all } i.$$

In other words $\{[w_i'', i] \mid i_X w'' = 0\}_i$ are the generators for cohomotopy. Since the corresponding element $(0, \ldots, 0, w_i'', 0, \ldots, 0)$ represents a cohomotopy class whenever $i_X w_i'' = 0$, Φ^* is injective.

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