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AN ENERGY ESTIMATE OF THE SOLUTION OF HARMONIC MAP HEAT EQUATION

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§1. Introduction

In this paper, we obtain an energy estimate of the solution of harmonic map heat equation. An estimate of this type is of fundamental importance in the analysis of the harmonic map heat equation.

The method of proof of our result is the maximum principle method modeled after the technique of Cheng [Ch] and Choi [C]. In their papers, they obtained the energy estimate of a harmonic map which is sharp enough to imply the Liouville theorem for harmonic map. However, their method cannot be directly applied to our situation because our case involves time also. Thus we have to modify the argument to fit our case. As a result, we were able to obtain the energy estimate of the solution of the harmonic map heat equation in terms of the initial energy density and various geometric quantities, assuming the target manifold have nonpositive sectional curvature. As a corollary, we can prove that the energy density of the solution of the harmonic map heat equation is bounded above by the energy density of the initial map times some constant expressible solely in terms of geometric bounds, when the solution of the harmonic map heat equation grows sublinearly, the domain manifold has nonnegative Ricci curvature, and target manifold have nonpositive sectional curvature.

§2. Notation

In this section, we introduce the notations and the formulas to be used through our paper. Let (M, g) and $(N, \overline{g})$ be two complete Riemannian manifolds, and let $u : M \times [0, \infty) \longrightarrow N$ be a smooth map. Choose an orthonormal frame $\{e_\alpha, \frac{\partial}{\partial t}\}$ in a neighborhood of $(x, t) \in M \times [0, \infty)$ and

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an local orthonormal frame $\{f_i\}$ in a neighborhood of $u(x,t) \in N$. Let $\{\theta_\alpha, dt\}$ and $\{\omega_i\}$ be the dual coframes of $\{e_\alpha, \frac{\partial}{\partial t}\}$ and $\{f_i\}$ respectively.

Denote $d = d_M + \frac{\partial}{\partial t}dt$ is the exterior derivative on $M \times [0,\infty)$ where d_M is the exterior derivative on M. Define $u_{i\alpha}$ and u_{it} by

$$u^*(\omega_i) = \sum_\alpha u_{i\alpha}\theta_\alpha + u_{it}dt.$$

We define the covariant derivatives, $u_{i\alpha\beta}$ and $u_{i\alpha t}$ by

$$\sum_\beta u_{i\alpha\beta}\theta_\beta + u_{i\alpha t}dt = du_{i\alpha} - \sum_j u_{j\alpha}u^*\omega_{ji} - \sum_\beta u_{i\beta}\theta_\beta\alpha.$$

Since $du_{i\alpha} = d_M u_{i\alpha} + u_{i\alpha t}dt$,

$$\sum_\beta u_{i\alpha\beta}\theta_\beta = d_M u_{i\alpha} - \sum_j u_{j\alpha}u^*\omega_{ji} - \sum_\beta u_{i\beta}\theta_{\beta\alpha}.$$

We define the energy density function $e(u)$ of u by $e(u)(x,t) = \sum_{i,\alpha} u_{i\alpha}^2$.

Suppose u is a solution of heat equation harmonic map. Then we have the following Bochner type formula:

$$\frac{1}{2}(\Delta - \frac{\partial}{\partial t})e(u) = \sum_{i,\alpha,\beta} u_{i\alpha\beta}^2 - \sum_{i,j,k,l,\alpha,\beta} R_{ijkl}u_{i\alpha}u_{j\beta}u_{k\alpha}u_{l\beta} + \sum_{\alpha,\beta,i} K_{\alpha\beta}u_{i\alpha}u_{i\beta}$$

where R_{ijkl} is the curvature tensor on N and $K_{\alpha\beta}$ is the Ricci tensor on M.

Consider the function $\rho^2(u(x,t))$ on $M \times [0,\infty)$ where ρ is a distance function in N from a fixed point $p \in N$. Since the sectional curvature of N is nonpositive and u is the solution of heat equation, we have

$$(2.1) \quad (\Delta - \frac{\partial}{\partial t})\rho^2(u) = (\rho^2)_i(u_{i\alpha\alpha} - u_{it}) + D^2(\rho^2)(u_*e_\alpha, u_*e_\alpha) \\ \geq 2e(u),$$

where the last inequality follows from the well known Hessian comparison theorem [W].

Fix a point $x_0 \in M$, let $\gamma(x)$ be the distance function from x_0 in M. Denote $B_a(x_0)$ to be the open geodesic ball with center x_0 and radius $a > 0$.

DEFINITION. A map $u : M \times [0, \infty) \to N$ is said to have sublinear growth if for any $T > 0$,

$$\limsup_{a \to \infty} \frac{\mu(u, a, T)}{a} = 0$$

where $\mu(u, a, T) = \sup\{\rho(u(x,t)) : (x,t) \in B_a(x_0) \times [0,T]\}$.

§3. Main Theorem

THEOREM. *Suppose that (M, g) is a complete Riemannian manifold with Ricci curvature bounded below by $-K \leq 0$ and $(N, \overline{g})$ is a complete Riemannian manifold with nonnegative sectional curvature. Let $u : M \times [0, \infty) \to N$ is a solution of heat equation for harmonic map with the initial map $h \in C^1(M, N)$ satisfying*

$$(1) \qquad (\Delta - \frac{\partial}{\partial t}) u(x,t) = 0 \qquad (x,t) \in M \times [0, \infty)$$

$$(2) \qquad u(x, 0) = h(x) \qquad x \in M.$$

Then for any $a > 0$, $x_0 \in M$ and $T > 0$,

$$\sup_{B_a(x_0) \times [0,T]} e(u) < C_4 \max\{\sup_M e(h),$$

$$K\mu^2(u,a,T) + \frac{(1+\sqrt{K})a\mu(u,a,T)}{a^2} + \frac{\mu(u,a,T)}{a^2} + \frac{\mu(u,a,T)}{a^2}\},$$

where $C_4 = C(K, a, n) > 0$.

Proof. Fix $x_0 \in M$, $a > 0$, and $T > 0$. Let p be a point in N, and let ρ denote the distance function from p in N. Choose a constant $b > 0$ such that $\mu(u, a, T) < b$. Consider the function

$$\Phi = \log\left\{\frac{(a^2 - \gamma^2)^2 e(u)}{(b^2 - \rho^2(u))^2}\right\},$$

which is defined on $B_a(x_0) \times [0, T)$. Since $\Phi = 0$ on $\partial B_a(x_0) \times [0, T]$, Φ achieves its maximum at some point in $[0, T] \times B_a(x_0)$. Let (x_1, t_1) be a point in $B_a(x_0) \times [0, T]$ such that

$$\Phi(x_1, t_1) = \max_{B_a(x_0) \times [0,T]} \log\left\{\frac{(a^2 - \gamma^2)^2 e(u)}{(b^2 - \rho^2(u))^2}\right\}.$$

Then there are two cases depending on whether $t_1 = 0$ or $0 < t_1 \le T$. First assume that $t_1 = 0$, then at any $(x,t) \in B_a(x_0) \times [0,T]$.

$$\begin{aligned}\frac{(a^2-\gamma^2)^2 e(u)}{(b^2-\rho^2(u))^2}(x,t) &\le \frac{(a^2-\gamma^2)^2 e(u)}{(b^2-\rho^2(u))^2}(x_1,t_1)\\ &\le \frac{a^4}{(b^2-\rho^2(u))^2}\sup_M e(h).\end{aligned}$$

Thus for $(x,t) \in B_a(x_0) \times [0,T]$,

$$e(u)(x,t) \le \frac{b^2}{(b^2-\rho^2(u))^2}\sup_M e(h). \tag{3.1}$$

Now we consider the case that $0 < t_1 \le T$, then Φ has the following properties:

$$\Delta\Phi(x_1,t_1) \le 0, \quad d_M\Phi(x_1,t_1) = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\Phi(x_1,t_1) \ge 0.$$

Rewriting these relations, we have at (x_1,t_1),

$$0 = \frac{-2d\gamma^2}{a^2-\gamma^2} + \frac{de(u)}{e(u)} + \frac{2d\rho^2(u)}{b^2-\rho^2(u)} \tag{3.2}$$

$$\begin{aligned}0 \ge{}& \frac{-2\Delta\gamma^2}{a^2-\gamma^2} + \frac{-2|d\gamma^2|^2}{(a^2-\gamma^2)^2} + \frac{(\Delta-\frac{\partial}{\partial t})e(u)}{e(u)}\\ &- \frac{|de(u)|^2}{e(u)^2} + \frac{2(\Delta-\frac{\partial}{\partial t})\rho^2(u)}{b^2-\rho^2(u)} + \frac{2|d\rho^2(u)|^2}{(b^2-\rho^2(u))^2}\end{aligned} \tag{3.3}$$

Applying the Bochner type formula and (2.1), we have, for $(x,t) \in B_a(x_0) \times [0,T]$,

$$\begin{aligned}0 \ge{}& \frac{-2\Delta\gamma^2}{a^2-\gamma^2} - \frac{2|d\gamma^2|^2}{(a^2-\gamma^2)^2} - \frac{1}{2}\frac{|de(u)|^2}{(e(u))^2} - 2K\\ &+ \frac{4e(u)}{b^2-\rho^2(u)} + \frac{2|d(\rho^2(u))|^2}{(b^2-\rho^2(u))^2}.\end{aligned} \tag{3.4}$$

Putting (3.2) into (3.4),

$$0 \geq \frac{-2\Delta\gamma^2}{a^2-\gamma^2} - \frac{4|d\gamma^2|^2}{(a^2-\gamma^2)^2} - \frac{4|d\gamma^2||d\rho^2(u)|}{(a^2-\gamma^2)(b^2-\rho^2(u))} - 2K + \frac{2e(u)}{b^2-\rho^2(u)}.$$

By the Gauss' lemma and Schwartz inequality, we get $|d\rho^2(u)| \leq \sqrt{e(u)}$. Then the Hessian comparison theorems (see [W]) implies that $\Delta\gamma^2 \leq C_1(1+\sqrt{K}\gamma)$ for some constant $C_1 > 0$.

Therefore

$$0 \geq -2K - \frac{2C_1(1+\gamma\sqrt{K})}{a^2-\gamma^2} - \frac{16\gamma^2}{(a^2-\gamma^2)^2} - \frac{8\gamma}{(a^2-\gamma^2)(b-\rho^2(u))}\sqrt{e(u)} + \frac{2e(u)}{b^2-\rho^2(u)}. \tag{3.5}$$

Since the (3.5) is a quadratic inequality for $\sqrt{e}(u)$ with positive leading coefficient, we have

$$e(u)(x_1,t_1) \leq 4\max\left\{\frac{128\gamma^2}{(a^2-\gamma^2)^2}, K(b^2-\rho^2(u)) + \frac{C_1(1+\gamma\sqrt{K})(b-\rho^2(u))}{(a^2-\gamma^2)} + \frac{8\gamma^2(b^2-\rho^2(u))}{(a^2-\gamma^2)^2}\right\}.$$

Therefore at any $(x,t) \in B_{\frac{a}{2}}(x_0) \times (0,T]$,

$$\begin{aligned}
&\left\{\frac{(a^2-\gamma^2)^2 e(u)}{(b^2-\rho^2(u))^2}\right\}(x,t) \\
\leq &\left\{\frac{(a^2-\gamma^2)^2 e(u)}{(b^2-\rho^2(u))^2}\right\}(x_1,t_1) \\
\leq &\, 4\max\left\{\frac{16a^2}{(b^2-\rho^2(u(x_1,t_1)))^2},\right. \\
&\left.\frac{Ka^2}{(b^2-\rho^2(u(x_1,t_1)))} + \frac{C_1(1+a\sqrt{K})a^2}{(b^2-\rho^2(u(x_1,t_1)))} + \frac{8a^2}{(b^2-\rho^2(u(x_1,t_1)))^2}\right\}
\end{aligned}$$

Thus

$$e(u)(x,t) \leq 4 \max \left\{ \frac{256a^2b^4}{9(b^2-\rho^2(u(x_1,t_1)))^2a^4}, \right.$$
$$\frac{16Ka^4b^4}{9(b^2-\rho^2(u(x_1,t_1)))a^4} + \frac{16C_1(1+\sqrt{K}a)a^2b^4}{9a^4(b^2-\rho^2(u(x_1,t_1)))}$$
$$\left. + \frac{128a^2b^4}{9a^4(b^2-\rho^2(u(x_1,t_1)))} \right\},$$

for $(x,t) \in B_{\frac{a}{2}}(x_0) \times (0.T]$.

Letting $b = \sqrt{2}\mu(u,a,T)$, we get

$$e(u)(x,t) \leq 4 \max \left\{ \frac{1024}{9a^2}, \frac{64}{9}K\mu^2(u,a,T) \right. \tag{3.6}$$
$$\left. + \frac{64C_1(1+\sqrt{K}a)\mu^2(u,a,T)}{9a^2} + \frac{512\mu^2(u,a,T)}{9a^2} \right\}.$$

Then by (3.1) and (3.6),

$$e(u)(x,t) \leq C_2 \max \left\{ \sup_M e(h), \frac{1}{a^2}, K'\mu^2(u,a,T) \right.$$
$$\left. + \frac{(1+\sqrt{K}a)\mu^2(u,a,T)}{a^2} + \frac{\mu^2(u,a,T)}{a^2} \right\}.$$

for any $(x,t) \in B_{\frac{a}{2}} \times [0,T]$.

The result below easily follows from the Main Theorem as a corollary.

COROLLARY. *Let (M,g) be a complete Riemannian manifold with nonnegative Ricci curvature and let $(N,\overline{g})$ be a complete Riemannian manifold with nonpositive sectional curvature. Let $u : M \times [0,\infty) \to N$ be a solution of heat equation for harmonic map with the initial map $h \in C^1(M,N)$. Suppose u has sublinear growth. Then*

$$\sup_{M\times[0,\infty)} e(u) \leq C_6 \sup_M e(h)$$

for some constant C_6 independent of h or u.

Proof. Since u has sublinear growth, there exists a sequence $\{a_i\}$ such that $a_i \to \infty$ and

$$\lim_{i\to\infty} \frac{\mu(u, a_i, T)}{a_i} = 0.$$

Therefore by replacing a with a_i and letting i goes to ∞, we get

$$e(u(x,t)) \leq C_6 \sup_M e(h) \qquad \text{for } (x,t) \in M \times [0.T].$$

Since T is arbitrary, our theorem is valid for all time.

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