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COMPLETE SURFACES WITH TOTAL CURVATURE

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1. Introduction

Let (M^2, g) be a complete finitely connected oriented Riemannian surface with total curvature $c(M)$. By total curvature, we mean the improper integral of the Gauss curvature over the Riemannian volume form. Such a space has the complete distance function induced from the arc length of curves. We denote it by d.

Many interesting facts are known about such surfaces. Among them, the first result was Cohn-Vossen inequality[CV]. Then it was improved as asymptotic Gauss-Bonnet type formulas mainly by Fiala[F] and Hartman[H]. Recently there has been more elaborate studies on the subject by Shiohama and others[S1,S2,S3,Sh1,Sh2,Y]. All these results can be summarized as saying that asymptotic geometry of such surfaces resembles that of Euclidean cones. In fact it is easy to see, from the work of Fiala, etc. that the family of metric spaces $(M, \lambda d)_{0<\lambda\leq 1}$ is a precompact family in Gromov-Hausdorff topology when the total curvature of M is finite and, therefore, it has some geometry in the limit. In this paper we will show that, for M above with finite total curvature, the family $(M, \lambda d)$ is actually convergent in Gromov-Hausdorff distance to a Euclidean cone as $\lambda \to 0^+$. On the other hand, an example shows that the finiteness condition on the total curvature is necessary. In [S4] Gromov-Hausdorff convergence is claimed for the case of infinite total curvature. This turns out to be wrong as we see below.

For simplicity we assume that our surface M has only one end. The same is true for surfaces with multiple ends and its proof is obvious from the one end case. Here we briefly explain the geometry of rays and large geodesic balls and refer the details to the exposition by Shiohama

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[S1]. Let p be a fixed point on M. The geodesic rays emanating from p forms a closed set $A(p)$. Given a real number $r > 0$ we consider the closed geodesic ball $\mathbf{B}(r)$ of radius r centered at p and its boundary curve $C(r) = \partial\mathbf{B}(r)$. We denote $A(p) \cap \mathbf{B}(r)$ by $D(r)$. Also let $E(r)$ denote the set of points on the minimizing geodesics from p to $\partial\mathbf{B}(r)$. It is known that $C(r)$ is actually homeomorphic to a circle for all r sufficiently large. Moreover it is piecewise smooth. The cut locus of p consists of several continuous curves diverging to infinity and these curves are disjoint from each other. The minimizing geodesics from each cut point to p bound a disk domain which is monotone increasing as the cut point moves along the cut locus toward infinity. The total width of such disk domains at radius r is asymptotically of order $o(r)$ as $r \to \infty$. Therefore the domains where there is no ray from p becomes arbitrarily thin, as one moves toward infinity, compared to the distance r from p.

A standard Hausdorff distance between two subsets A, B of a compact metric space Z is defined as

$$d_H^Z(A,B) = \inf\{\epsilon \mid \mathbf{B}(A,\epsilon) \supset B, \mathbf{B}(B,\epsilon) \supset A\},$$

where $\mathbf{B}(A,\epsilon)$ denotes the closed ϵ-neighborhood of A. Let X, Y be compact metric spaces. Then the Gromov-Hausdorff distance between X, Y is

$$d_{GH}(X,Y) = \inf d_H^Z(X,Y),$$

where the infimum is over all compact metric spaces Z and all isometric embeddings of X, Y into Z.(See [GLP])

Let (X_i, p_i) be a sequence of (not necessarily compact) pointed, complete, locally compact, inner metric spaces. We say that this sequence converges to (X, p) of the same kind in *Gromov-Hausdorff topology* if, for each r, $\mathbf{B}^i(p_i, r)$ converges in Gromov-Hausdorff distance to $\mathbf{B}(p, r)$, where $\mathbf{B}^i$ denotes the closed ball in X_i.(Cf. [GLP]) Now the statement of our main theorem is

THEOREM A. *Let M be a surface defined as above with finite total curvature and only one end. Let p be a fixed point in M. Then the family $(M, p, \lambda d), 0 < \lambda \leq 1$ is a precompact family and converges in the Gromov-Hausdorff topology to a Euclidean cone over a circle of length $2\pi\chi(M) - c(M)$ as $\lambda \to 0^+$.*

According to Gromov[G], a family $\mathcal{C}$ of compact metric spaces is precompact in Gromov-Hausdorff distance if and only if, for every $\epsilon > 0$, there is a positive integer $N(\epsilon)$ such that the minimum number of ϵ-balls needed to cover each X in $\mathcal{C}$ is bounded above by $N(\epsilon)$.

Example. Let (Δ, Hd) be the Poincaré disk with Gaussian curvature -1, where Hd is its hyperbolic distance function. Then $c(\Delta, Hd) = -\infty$. Consider the family $\Delta_\lambda = (\Delta, Hd_\lambda)$, $0 < \lambda \leq 1$ of complete metric spaces, where $Hd_\lambda = \lambda Hd$. Each Δ_λ is a Riemannian surface with constant Gaussian curvature $-1/\lambda^2$. We define the standard geodesic polar coordinate system $\psi_\lambda : (\Delta_\lambda, O) \to (\mathbb{R}^+ \cup 0) \times S^1$. Then $Hd_\lambda(\psi_\lambda^{-1}(1,\theta), \psi_\lambda^{-1}(1,\varphi)) \to 2$ as $\lambda \to 0^+$ for every $\theta \neq \varphi$. Now one deduce from this that the minimum number of δ-balls ($\delta < (1/2)$) needed to cover each unit ball in Δ_λ is not uniformly bounded for all λ. Thus Δ_λ is not a precompact family. In fact, same estimate shows that, if $\lim \lambda_n = 0$, no sequence $\{\Delta_{\lambda(n)}\}$ is precompact and not convergent in Gromov-Hausdorff distance. Thus no rescaling-down of Poincaré disk is convergent in Gromov-Hausdorff topology.

2. Proof of Theorem A

When we say total curvature exists as an improper integral on M, we mean that, for an arbitrary compact exhaustion A_i(i.e., $\cup A_i = M$), $\lim c(A_i)$ exists and the limits coincide for all such exhaustions. The existence of total curvature, then, guarantees the convergence of

$$\int_M G^+ dV_M \quad \text{and} \quad \int_M G^- dV_M,$$

where $G^+ = \max\{G, 0\}$ and $G^- = \max\{-G, 0\}$, and moreover the total positive curvature must be finite.[CV] Therefore we may assume that the total absolute curvature is convergent.(It may either be finite or infinite.)

Now suppose that the total Gauss curvature of M is finite. Let $0 < \epsilon < R$ and $\lambda > 0$ be real numbers. Consider a domain

$$\Omega_\lambda = \{x \mid \frac{\epsilon}{\lambda} < d(x,p) < \frac{R}{\lambda}\} \subset M,$$

which is diffeomorphic to an annulus. Boundary of Ω_λ consists of two piecewise smooth Jordan curves $C(\epsilon/\lambda)$ and $C(R/\lambda)$. Using the minimizing geodesics in $E(R/\lambda)$ one gets a geodesic coordinate system on

$E(R/\lambda)$. Now there is $\eta > 0$ so that whenever $0 < \lambda < \eta$, $\Omega_\lambda - D(R/\lambda)$ is sufficiently thin compared to $1/\lambda$ and the total curvature on this domain is also negligible. Therefore in the arguments below we will neglect the existences of such thin strips in Ω_λ. We will denote the Riemannian metric in this coordinate system as

$$g = dr^2 + f(r,\theta)^2 d\theta^2, \quad \frac{\epsilon}{\lambda} \le r \le \frac{R}{\lambda}, \quad \theta \in J_\lambda,$$

where J_λ is a disjoint union of finitely many closed intervals in S^1 and is monotone decreasing as $\lambda \to 0^+$ so that $\cap_\lambda J_\lambda$ matehes exactly with $A(p)$. Then the Gauss curvature function is given as $G = -f_{rr}/f$. Now by rescaling Ω_λ by the factor λ one gets the domain ω_λ in $(M, \lambda d)$, on which the rescaled metric is given, in geodesic polar coordinate, as

$$g_\lambda = \lambda g = dr^2 + \lambda^2 f(\frac{r}{\lambda}, \theta)^2 d\theta^2$$

for $\epsilon \le r \le R$ and $\theta \in J_\lambda$. Obviously the total curvature of g_λ on ω_λ coincides with that of g on Ω_λ.

We will show that the rescaled metric coefficients $f_\lambda(r,\theta) = \lambda f(r/\lambda, \theta)$ behave nicely so that the limiting distance structure is a flat one. Now on any domain $K = (a,b) \times J$, we have

$$\begin{aligned} \int_K |G_\lambda| dV_{M_\lambda} &= \int_J \int_a^b \frac{|f_{\lambda;rr}|}{f_\lambda} f_\lambda \, dr d\theta \\ &\ge \int_J |f_{\lambda;r}(a) - f_{\lambda;r}(b)| \, d\theta. \end{aligned}$$

Let $\delta > 0$ be given. Since $|G|$ is integrable, by choosing η above sufficiently small, the integral above can be made less than δ for all $a, b > \epsilon$. That is, the function $|f_{\lambda;r}(a) - f_{\lambda;r}(b)|$ is L^1-convergent to 0 in $d\theta$. Therefore it also converges to 0 in measure, i. e., it is uniformly close to 0 except on a set which can be chosen arbitrarily small for all $\lambda < \eta$. Since the estimate is independent of $a, b\,(> \epsilon)$, $f_{\lambda;r}$ is uniformly close to $f_{\lambda;r}(\epsilon)$ except on some intervals of θ of which the total length in $d\theta$ is also small. Denote the union of these intervals by I and let $\int_I f_\lambda(\epsilon,\theta) d\theta < \epsilon_1$ for given $\epsilon_1 > 0$ by choosing η small and, therefore, I also small. Since the total curvature on the strips $S = [\epsilon, R] \times I$ is almost 0, the total width of the strips becomes negligible. To see this, one observes that

$$\int_S G = \int_\epsilon^R \int_I -f_{\lambda;rr}\, d\theta dr = F_{\lambda;r}(\epsilon) - F_{\lambda;r}(R),$$

where $F_\lambda(r) = \int_I f_\lambda(r,\theta)d\theta$. Since this integral is small uniformly in r, $F_\lambda(r)$ is uniformly close to a linear function $\epsilon_1 r$ and, hence, the total width in K for the strips I is small.

Now except on S, $f_\lambda(r)$ is uniformly close to $f_\lambda(\epsilon)r$ for $\epsilon < r < R$. Finally, we show the Gromov-Hausdorff convergence for the rescaling.

GROMOV'S CONVERGENCE LEMMA. *Let X_i and X be metric spaces of diameters uniformly bounded above. Suppose, for each $\epsilon > 0$, there exist an ϵ-net N of X and ϵ-nets N_i of X_i, so that each N_i is in one-to-one correspondence with N and distance functions on N_i converge uniformly to that of N under the correspondence.*

Let $\epsilon_2 > 0$ be given. We choose an $\epsilon_2/4$-net N_r on $(0,\infty)$ and an $\epsilon_2/4$-net N_θ on $C(R) \cap A(p)$. Consider the net $N = N_r \times N_\theta$. Then $N \cap D(R)$ gives an $\epsilon_2/4$-net on $D(R)$. Now $N_\lambda = ((1/\lambda)N_r) \times N_\theta$ is an ϵ_2-net on Ω_λ for $\lambda < \eta$. Now since $f_\lambda(r,\theta)$ is also uniformly close to $f_\lambda(\epsilon,\theta)r$ on $[\epsilon, R] \times S^1$ except on very thin and straight strips, $d_\lambda((r_1/\lambda,\theta_1),(r_2/\lambda,\theta_2))$ is close to $((r_1)^2 + (r_2)^2 - r_1 r_2 \cos\alpha)^{1/2}$ as $r \to +\infty$, where α = limit of total curvature of $C(r)$ between θ_1 and θ_2. The limit distance on N obviously gives an ϵ_2-net for a Euclidean cone with cone angle $2\pi - c(M)$. This completes the proof.

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