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CONTINUOUS EXTENSIONS OF TOPOLOGICAL MODULE ENDOMORPHISMS

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In this short article, we study the extensions of continuous endomorphisms of a topological module that are given by scalar multiplication by a ring element. First, let us briefly check the details in the case of the circle, which gives the motivation for the generalization. Let S^1 be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ that is identified with a topological abelian group $X = \mathbb{R}/\mathbb{Z}$. It is also identified with the half-open interval $[0,1)$. The continuous group-homomorphisms of S^1 into itself are given by the formula $h : z \mapsto z^m$ for some integer m, where the integer m is the winding number of h and $|m|$ is the number of elements in the kernel of h if $m \neq 0$. There are several ways to prove this fact and the following argument is not easily found elsewhere: We lift the $\mathbb{Z}$-module endomorphism $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to a continuous $\mathbb{Z}$-module endomorphism $\phi : \mathbb{R} \to \mathbb{R}$ by the formula $h(z) = \exp(2\pi i \phi(x))$ for $z = e^{2\pi i x} \in \mathbb{R}/\mathbb{Z}$. Then we may assume that $\phi(0) = 0$ and $\phi(x+1) = \phi(x) + m$ for some integer m. Now we prove that $h(z) = z^m$. Since the map $z \mapsto z^m$ is a homomorphism of the circle, we see that $g(z) = h(z)\, z^{-m}$ is also a continuous homomorphism. Hence it suffices to show that g is the identity map. Note that $g(e^{2\pi i x}) = \exp(2\pi i \psi(x))$ where $\psi(x) = \phi(x) - mx$ and $\psi(x+1) = \psi(x)$ for every $x \in \mathbb{R}$. From the fact that h is a homomorphism, we observe that $\psi(x+y) = \psi(x) + \psi(y) + d(x,y)$ where $d(x,y)$ is an integer depending on x, y. Since ψ is a continuous function on $\mathbb{R}$, we conclude that d is an integer-valued function which is continuous with respect to x and y, thus it is constant. By substituting $x = 0$ and $y = 0$ we have $\psi(x+y) = \psi(x) + \psi(y)$, i.e., $\psi : \mathbb{R} \to \mathbb{R}$ is an additive group homomorphism. Suppose that there exists $x_0 \in \mathbb{R}$ for which $\psi(x_0) \neq 0$. Then $\{\psi(nx_0) : n \in \mathbb{Z}\}$ is an unbounded set in $\mathbb{R}$, which

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contradicts the boundedness of the periodic function ψ. Therefore $\psi \equiv 0$ and $\phi(x) = mx$.

Now we look at the same fact from the point of linear algebra. If we regard S^1 as a $\mathbb{Z}$-module, the group-homomorphism $h : S^1 \to S^1$ becomes a $\mathbb{Z}$-module homomorphism $h : X \to X$. Hence the problem is to show that there exists an integer m satisfying $h(x) = mx$, $x \in X$. If X were a one dimensional module over the ring $\mathbb{Z}$, then there would exist a 1×1 matrix m associated with a $\mathbb{Z}$-linear module homomorphism as in the case of vector space linear transformations. But X is infinitely generated and a straightforward application of linear algebraic idea would not work. We approximate X by submodules which are finitely generated. Consider the submodule $X_d = \{0, \frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d}\}$ which is generated by one element $\frac{1}{d}$. If d is sufficiently large, then X_d approximates X in a sense. Note that $d \cdot h(\frac{1}{d}) = h(0) = 0$ implies that $h(\frac{1}{d}) = \frac{k}{d}$ for some k, hence $h(X_d) \subset X_d$. Thus we may define $h_d : X_d \to X_d$ as the restriction of h. We expect that h_d would approximate the continuous mapping h if d is sufficiently large. For a trivial endomorphism h, there is nothing to prove. So we assume that h is nontrivial. Since $\frac{1}{d}$ generates X_d, the discrete mapping h_d is determined by the formula $h_d(\frac{1}{d}) = n \cdot \frac{1}{d}$ for some n, $0 < n < d$. Similarly, we define X_{2d} and h_{2d}. Then h_{2d} is determined by the formula $h_{2d}(\frac{1}{2d}) = m \cdot \frac{1}{2d}$ for some m, $0 < m < 2d$. Note that the restriction of h_{2d} on X_d is h_d, hence $2 \cdot h_{2d}(\frac{1}{2d}) = h_{2d}(\frac{1}{d}) = h_d(\frac{1}{d})$, $2 \cdot \frac{m}{2d} = \frac{n}{d}$, $d | m - n$ and hence $m = n + kd$ for some integer k. Thus we may conclude that either $m = n$ or $m = n + d$. The latter corresponds to the case when the winding number of h is negative. Replacing h by $-h$ if necessary, we may assume that $m = n$. Continuing the argument indefinitely, we see that h is nothing but the multiplication by the 1×1 matrix n on the dense submodule consisting of the elements $k/2^j$, $j \in \mathbb{N}$, $0 \leq k < 2^k$. Since h is continuous, it is represented by the same matrix on the whole module $\mathbb{R}/\mathbb{Z}$. This is a roundabout way of seeing things but it answers the question.

Now we apply the same idea for general cases. For the various definitions of algebraic structures we refer to [1]. Let M be a module over a commutative ring R with identity and h an R-module endomorphism of M, that is, $h(x_1 + x_2) = h(x_1) + h(x_2)$ for $x_1, x_2 \in M$ and $h(rx) = r\,h(x)$ for $r \in R$, $x \in M$. If M is generated by finitely many elements $\{v_1, v_2, \ldots, v_d\}$ in M, then every element $x \in M$ is expressed as a sum $x = r_1v_1 + \cdots + r_dv_d = \sum_{j=1}^{d} r_jv_j$, $r_j \in R$. The coefficients r_i's are not unique in general if M is not *torsion-free*, that is, $nx \neq 0$ for $n \neq 0$, $0 \neq x \in M$. Note that $h(v_j) = \sum_{i=1}^{d} a_{ij}v_i$ for some $a_{ij} \in R$, hence $h(x) = \sum_{j=1}^{d} \sum_{i=1}^{d} a_{ij}r_jv_j$ and we say that h is represented by a matrix (a_{ij}). If M is generated by one element, then h is represented by 1×1 matrix $a \in R$, in other words, $h(x) = ax$ for every $x \in M$. For infinitely generated modules there is no representation of endomorphisms as above.

Recall that a ring R is said to be a *topological ring* if it is a topological space and all the ring operations are continuous. A module M is called a *topological module* over a topological ring R if all the module operations are continuous. If a topological module M is compact, then it might not be torsion-free hence it would be usually impossible to find a unique matrix representation even if M is finitely generated. A metric ρ on M is said to be *translation-invariant* if $\rho(u + x, u + y) = \rho(x, y)$ for every $u, x, y \in M$. Now we have the following

PROPOSITION. *Let M_1 be a dense submodule of a topological module M over a topological ring R with identity and let ρ be a translation-invariant metric on M. If a continuous R-module homomorphism $h : M_1 \to M_1$ is represented by a scalar multiplication, i.e., there exists $a \in R$ such that $h(x) = ax$ for every $x \in M_1$, then h can be extended onto M uniquely as a continuous R-module homomorphism and is also represented by the same a, that is, $h(x) = ax$ for every $x \in M$.*

Proof. Note that the translation-invariance implies the uniform continuity of h on the metric space M_1. Hence it has a unique continuous extension g onto its closure M. Take an arbitrary element $x \in M$. Then there exists a sequence $x_n \in M_1$ that converges to x. Since the module operations are continuous, we have that $g(x) = \lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} ax_n = a \lim_{n\to\infty} x_n = ax$.

REMARK. A typical application of the above result is to approximate an infinitely generated topological module M by an increasing sequence of finitely generated submodules $M_1 \leq M_2 \leq \cdots \leq M_k \leq \cdots \leq M$ satisfying the conditions that (i) $h(M_k) \leq M_k$ for every k, (ii) h restricted on M_k is given by the same ring element a for every k and (iii) $\cup_k M_k$ is dense in M.

References

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