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EXISTENCE OF SOLUTIONS FOR PSEUDO-LAPLACIAN EQUATION WITHOUT GROWTH CONDITION

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1. Introduction

In this paper we deal with the solvability of the Dirichlet problem for the Pseudo-Laplacian equation $-\Delta_p u = f(u) + h(x)$, where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, in the case where no growth condition is imposed on the nonlinearity of f. For $p = 2$, the solvablity was proved by D. Figueiredo and J. Gossez [2], we generalize their result to arbitrary $p \geq 2$.

Let Ω be a bounded and open subset of R^N, $N \geq 2$ and the function $f : R \to R$ be continuous. We shall consider the existence of solutions for the Dirichlet problem

$$\text{(1)} \qquad \begin{aligned} \text{-div}(|\nabla u|^{p-2}\nabla u) &= f(u) + h(x) \quad \text{in } \Omega \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

where h is a given function on Ω. Let $F(s) = \int_0^s f(t)dt$ satisfy the condition of non-résonance

$$\text{(2)} \qquad \lim_{s\to\pm\infty} \sup \frac{pF(s)}{s^p} < \lambda_1$$

where λ_1 is the first eigenvalue of $-\Delta_p u = \lambda |u|^{p-2}u$ on $W_0^{1,p}(\Omega)$. Our main result is the following:

THEOREM 1. *We suppose that the condition (2) be satisfied. Then for each $h \in L^\infty(\Omega)$, there is $u \in W_0^{1,p}(\Omega)$ such that $f(u) \in L^1(\Omega)$, $f(u)u \in L^1(\Omega)$ and u satisfies (1) in the following sense;*

$$\text{(3)} \qquad \int_\Omega |\nabla u|^{p-2} < \nabla u, \nabla v > = \int_\Omega f(u)v + \int_\Omega hv$$

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for all $v \in W_0^{1,p}(\Omega) \bigcap L^\infty(\Omega)$ *and for* $v = u$.

The above theorem is given by an immediate consequence of four propositions which are proved in Section 2. The proofs of these four propositions are based on the existence theorem of the following quasilinear elliptic equations and J.Webb's theorem [5]. First, we recall the existence theorem of the quasilinear elliptic equation.

In the quasilinear Dirichlet problem

$$\begin{aligned} &\text{-div}(|\nabla u|^{p-2} \nabla u) = f(x,u) \quad \text{in } \Omega \\ &u|_{\partial\Omega} = 0 \end{aligned} \tag{4}$$

where Ω is a C^1 bounded domain of R^N and $f : \Omega \times R \to R$ is a Carathéodory function. Let us define a function $F : \Omega \times R \to R$ by

$$F(x,u) = \int_0^u f(x,s)ds \tag{5}$$

and let $\lambda_1 > 0$ be the first eigenvalue of the problem

$$\begin{cases} -\Delta_p u &= \lambda|u|^{p-2}u \\ u|_{\partial\Omega} &= 0. \end{cases} \tag{6}$$

Let q be such that

$$\begin{cases} 1 < q < \frac{Np}{N-p} & \text{if } \ 1 < p < N \\ 1 < q < \infty & \text{if } \ p \geq N \end{cases}$$

and r be the conjugate exponent of q. In order to solve the problem (4) in the Sobolev space $W_0^{1,p}(\Omega)$, we assume that there exist $a \geq 0$ and $b(x) \in L^r(\Omega)$ such that

$$|f(x,u)| \leq a|u|^{q-1} + b(x). \tag{7}$$

We also assume that there is $\alpha(x) \in L^\infty(\Omega)$ such that

$$\lim_{|u|\to\infty} \sup \frac{pF(x,u)}{u^p} \leq \alpha(x) < \lambda_1 \tag{8}$$

uniformly for almost all $x \in \Omega$. Then the given Dirichlet problem has at least one solution u in $W_0^{1,p}(\Omega)$.(cf. [3], [4])

2. Main Results

Using the above existence theorem and J.Webb's theorem [5], we will prove the following four propositions.

PROPOSITION 1. *We suppose that*

$$\inf_{s\geq 0} f(s) = -\infty, \quad \sup_{s\leq 0} f(s) = +\infty \tag{9}$$

Then for each $h \in L^\infty(\Omega)$, there is $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2} < \nabla u, \nabla v >= \int_\Omega f(u)v + \int_\Omega hv$$

for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and for $v = u$.

Proof. By the condition (9), there is $a < 0 < b$ such that $f(a) \geq \|h\|_\infty$ and $f(b) \leq -\|h\|_\infty$. Put

$$\hat{f}(s) = \begin{cases} f(s) & \text{for } s \in [a,b] \\ f(a) & \text{for } s < a \\ f(b) & \text{for } s > b. \end{cases}$$

Then the Dirichlet problem

$$\begin{aligned} -\Delta_p u &= \hat{f}(u) + h(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{10}$$

admits the solution u (in the weak sense) since $\hat{f}(s)$ satisfies the conditions (7) and (8). Then we claim that $a \leq u(x) \leq b$ for almost all $x \in \Omega$. To show the second inequality, Put

$$u_b(x) = \begin{cases} u(x) & \text{if } u(x) \leq b \\ b & \text{if } u(x) > b \end{cases}$$

and let $w = u - u_b$. It follows from (10) that

$$\int_\Omega |\nabla u|^{p-2} < \nabla u, \nabla w >= \int_\Omega (\hat{f}(u) + h)w.$$

Each integral holds over $\{x \in \Omega; u(x) > b\}$, a domain in which

$$\nabla u = \nabla w, \ \hat{f}(u) + h \leq 0, \ w \geq 0.$$

Hence $\int_\Omega |\nabla w|^p dx \leq 0$. Therefore $w = 0$ for almost all $x \in \Omega$. By the same method, the first inequality holds. Thus $a \leq u(x) \leq b$ for almost all $x \in \Omega$. Hence u is the solution of

$$\begin{cases} -\Delta_p u &= f(u) + h(x) \quad \text{in } \Omega \\ u|_{\partial\Omega} &= 0. \end{cases}$$

PROPOSITION 2. *We suppose that the condition (2) be satisfied and*

$$(11) \qquad \inf_{s \geq 0} f(s) > -\infty, \qquad \sup_{s \leq 0} f(s) < +\infty \quad (\frac{1}{p} + \frac{1}{p'} = 1).$$

Then for each $h \in W^{-1,p'}(\Omega)$, there is $u \in W_0^{1,p}(\Omega)$ such that $f(u) \in L^1(\Omega)$, $f(u)u \in L^1(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2} < \nabla u, \nabla v >= \int_\Omega f(u)v + \int_\Omega hv$$

for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and for $v = u$.

Proof. It follows from (2), (11) that there exist a constant C and $\lambda < \lambda_1$ such that

$$\begin{aligned} &f(s) \leq \lambda |s|^{p-2}s + C \quad s \geq 0 \\ &f(s) \geq \lambda |s|^{p-2}s - C \quad s \leq 0 \\ &\sup_{|s| \leq t} |f(s)| \in L^1(\Omega), \quad 0 \leq t < \infty \end{aligned}$$

The problem is equivalent to

$$(-\Delta_p - \lambda |u|^{p-2})u = f(u) - \lambda |u|^{p-2}u + h(x) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Put

$$T(u) = -\Delta_p u - \lambda |u|^{p-2}u$$

$$g(s) = \lambda |s|^{p-2}s - f(s)$$

Then the function $g(s)$ satisfies $g(s)$ sign $s \geq -C$. We note that each function which satisfies $g(s)$ sign $s \geq -C$ can be written as $g(s) = g_1(s) + g_2(s)$ where $g_1(s)$ and $g_2(s)$ are Carathéodory functions such that $g_1(s)s \geq 0$, $|g_2(s)| \leq 2C$ for $s \in R$. Indeed, we can take

$$g_1(s) = \begin{cases} (\ g(s) - g(0))^+ & s \geq 0 \\ -(g(s) - g(0))^- & s \leq 0 \end{cases}$$

$$g_2(s) = \begin{cases} -(g(s) - g(0))^- + g(0) & s \geq 0 \\ (g(s) - g(0))^+ + g(0) & s \leq 0. \end{cases}$$

Morever, $\sup\limits_{|s| \leq t} |g(s)| = h_t(s) \in L^1(\Omega)$. Furthermore, T is coercive for all $\lambda < \lambda_1$. Indeed,

$$\begin{aligned} < Tv, v > &= < (-\Delta_p - \lambda_1 |v|^{p-2})v, v > + < (\lambda_1 - \lambda)|v|^{p-2}v, v > \\ &= (1 - \frac{\lambda_1 - \lambda}{\lambda_1})(< -\Delta_p v, v > - < \lambda_1 |v|^{p-2}v, v >) \\ &+ \frac{\lambda_1 - \lambda}{\lambda_1} < -\Delta_p v, v > \geq \frac{\lambda_1 - \lambda}{\lambda_1} < -\Delta_p v, v > \\ &\geq \frac{\lambda_1 - \lambda}{\lambda_1} \| \nabla v\|^p_{W_0^{1,p}(\Omega)} \end{aligned}$$

The map $T : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ $(1 < p < \infty, \frac{1}{p} + \frac{1}{p'} = 1, p \geq 1)$ satisfies the three conditions of J.Webb's theorem with

$$A_i(x; \xi_0, \ldots, \xi_M) = |\xi_i|^{p-1} \ \text{sign}\ \xi_i$$

$$A_0(x; \xi_0, \ldots, \xi_M) = \lambda |u|^{p-1} \ \text{sign}\ u.$$

Also the function $g(x,t) : \Omega \times R \to R$ is a Carathéodory function satisfying the two conditions of J.Webb's theorem. Thus by [5] , for every $h \in W^{-1,p'}(\Omega)$, there exists $u \in W_0^{1,p}(\Omega)$ such that $g(\cdot, u) \in L^1(\Omega)$, $g(\cdot, u)u \in L^1(\Omega)$ and

$$< Tu, v > + \int g(x,u)v = \int fv$$

for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and for $v = u$.

PROPOSITION 3. *We suppose that the condition (2) be satisfied at -∞ and*

$$\inf_{s\geq 0} f(s) = -\infty, \quad \sup_{s\leq 0} f(s) < \infty. \tag{12}$$

Then for each $h \in L^r(\Omega)$ which is bounded above (where $r > 1$ if $N \leq p$, $r = \frac{Np}{N-p}$ if $2 < p < N$), there is $u \in W_0^{1,p}(\Omega)$ which is bounded above such that $f(u) \in L^1(\Omega)$, $f(u)u \in L^1(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2} < \nabla u, \nabla v >= \int_\Omega f(u)v + \int_\Omega hv$$

holds for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and for $v = u$.

Proof. By the condition (12), there is $b > 0$ such that $f(b) \leq -\sup\ h$. Put

$$\hat{f}(s) = \begin{cases} f(s) & s \leq b \\ f(b) & s > b. \end{cases}$$

It follows from Proposition 2 that there is $u \in W_0^{1,p}(\Omega)$ such that $\hat{f}(u) \in L^1(\Omega)$, $\hat{f}(u)u \in L^1(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2} < \nabla u, \nabla v >= \int_\Omega \hat{f}(u)v + \int_\Omega hv \tag{13}$$

for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and for $v = u$. We show that $u(x) \leq b$ for almost all $x \in \Omega$. Put

$$T = -\Delta_p u - h \in W^{-1,p'}.$$

Then, by (13), $< T, \varphi >= \int_\Omega \hat{f}(u)\varphi$ for each $\varphi \in \mathcal{D}(\Omega)$ where $< \, , >$ represents the duality between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. Thus $T = \hat{f}(u)$ in the sense of distribution. Put

$$u_b(x) = \begin{cases} u(x) & \text{if} \quad u(x) \leq b \\ b & \text{if} \quad u(x) > b. \end{cases}$$

and let $w = u - u_b$. Then $\hat{f}(u)w \in L^1(\Omega)$. Indeed

$$\hat{f}(u)w = 0 \quad \text{in} \quad \{x \in \Omega : u(x) = 0\},$$

$$|\hat{f}(u)w| = |\hat{f}(u)u|\frac{|w|}{|u|} \leq |\hat{f}(x,u)u| \in L^1(\Omega) \ \text{ in } \{x \in \Omega : u(x) \neq 0\}.$$

It follows from the theorem of Brezis-Browder [1] that we have

$$< T, w >= \int_\Omega \hat{f}(u)w$$

$$\text{i.e,} \quad \int_\Omega | \nabla u|^{p-2} < \nabla u, \nabla w >= \int_\Omega \hat{f}(u)w + \int_\Omega hw.$$

Each integral holds over $\{x \in \Omega : u(x) > b\}$, domain in which $\nabla u = \nabla w$, $\hat{f}(u) + h \leq 0$, $w \geq 0$. Hence $\int_\Omega | \nabla w|^p \leq 0$. Therefore $w = 0$ for almost all $x \in \Omega$. Thus $u(x) \leq b$ for almost all $x \in \Omega$. Hence, by Proposition 2, u is the solution of

$$\begin{cases} -\Delta_p u &= f(u) + h(x) \ \text{ in } \ \Omega \\ u|_{\partial\Omega} &= 0. \end{cases}$$

PROPOSITION 4. *We suppose that the condition (2) be satisfied at* $+\infty$ *and*

$$\inf_{s \geq 0} f(s) > -\infty, \quad \sup_{s \leq 0} f(s) = +\infty.$$

Then for each $h \in L^r(\Omega)$ *which is bounded below (where* $r > 1$ *if* $N \leq p$, $r = \frac{Np}{N-p}$ *if* $2 < p < N$*), there is* $u \in W_0^{1,p}(\Omega)$ *which is bounded below such that* $f(u) \in L^1(\Omega)$, $f(u)u \in L^1(\Omega)$ *and*

$$\int_\Omega | \nabla u|^{p-2} < \nabla u, \nabla v >= \int_\Omega f(u)v + \int_\Omega hv$$

holds for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *and for* $v = u$.

Proof. This proof is essentially the same as the Proposition 3.

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